REED COLLEGE
PHYSICS SEMINAR SCHEDULE

ALL SEMINARS ARE HELD AT 4:10 ON WEDNESDAY IN ROOM P123 UNLESS OTHERWISE INDICATED.

<table>
<thead>
<tr>
<th>Date</th>
<th>Speaker(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>February 3</td>
<td>MAYHEW, TUFILLARO*</td>
</tr>
<tr>
<td>SPECIAL</td>
<td>February 4, Thursday, VLH RICHARD SRAMEK, NRAO, &quot;RADIO SUPERNOVAE&quot;</td>
</tr>
<tr>
<td></td>
<td>February 10 FITZSIMMONS, OWEN</td>
</tr>
<tr>
<td></td>
<td>February 17 BENACQUISTA, WIENER</td>
</tr>
<tr>
<td>SPECIAL</td>
<td>February 18, Thursday PETER OLDS, Reed Chemistry, &quot;ATMOSPHERIC TRACE</td>
</tr>
<tr>
<td></td>
<td>GASES&quot;**</td>
</tr>
<tr>
<td></td>
<td>February 24 BONAR, MCNULTY</td>
</tr>
<tr>
<td></td>
<td>March 3 LITT, McGEHEE</td>
</tr>
<tr>
<td>SPECIAL</td>
<td>March 4, Thursday, VLH, 8:00 P.M. STEVEN MORSE, INTEL, &quot;HISTORY OF THE</td>
</tr>
<tr>
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<td>INTEL MICROPROCESSOR**</td>
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<tr>
<td></td>
<td>March 10 HENLEY, WEDELL</td>
</tr>
<tr>
<td>SPRING RECESS</td>
<td></td>
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<tr>
<td>March 24</td>
<td>MCCORMICK, WRIGHT</td>
</tr>
<tr>
<td>March 31</td>
<td>HELEN QUINN, SLAC, A.A. Knowlton Visiting Lecturer: Technical seminar;</td>
</tr>
<tr>
<td></td>
<td>topic to be announced</td>
</tr>
<tr>
<td>SPECIAL</td>
<td>March 31, 8:00 P.M. VLH HELEN QUINN, General lecture; topic to be</td>
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<tr>
<td></td>
<td>announced.</td>
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<tr>
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<td>April 7 ROBINSON, RUF</td>
</tr>
<tr>
<td>SPECIAL</td>
<td>April 13, Tuesday, 8:00 P.M., VLH WOODRUFF SULLIVAN III, University of</td>
</tr>
<tr>
<td></td>
<td>Washington, Harlow Shapley Visiting Lecturer: &quot;THE SEARCH FOR EXTRATER</td>
</tr>
<tr>
<td></td>
<td>RESTRIAL INTELLIGENCE—A PLAN FOR ACTION!</td>
</tr>
<tr>
<td></td>
<td>April 14, VLH WOODRUFF SULLIVAN III, &quot;THE AGE AND SIZE OF THE UNIVERSE&quot;</td>
</tr>
<tr>
<td></td>
<td>April 21 PAGLIN, SUTTON</td>
</tr>
<tr>
<td></td>
<td>April 28 CALVERT, GRONKE</td>
</tr>
</tbody>
</table>

**Another in the series "Summer Research Activities—1981"

*Unless otherwise indicated all speakers are Reed physics seniors.
Physics Seminar  Reed College  3/2/82

Title: Smiling Swingers or 4012 Physics.

Good afternoon. I intend to describe a system from my notebook of Classical Mechanics problems which on the surface looks simple, like a toy at best, but whose behavior is astonishingly complex and presents facets of more than academic luster. It is a problem concerning the orbits or trajectories of a particle in two-degrees of freedom under a very particular non-central force; to wit, the problem of two pendula. Now, the difficulty of this problem should really surprise none; for the general problem of motion of a classical particle in two degrees of freedom is nowhere near being solved. As the Russian mathematician V. I. Arnold comments in his 1980 book on Classical Mechanics:

"Analyzing a general potential system with two degrees of freedom is beyond the capability of modern science."

The problem I wish with you, the two pendula problem (t.p.p.), it consists of describing the motions of the following machine: (go to board; do not pass go, do not collect $200)

Two Pendula Machine

![Diagram of two pendula machine]

Given: at time $t=0$
- $r(0) = l$, $\theta(0) = \theta_0$, $m \in \mathbb{M}$
- $r(0) = \theta_0$, $\dot{\theta}(0) = \dot{\theta}_0$

Find: What happens? $\theta(t)$, $r(t)$, $t > 0$
The machine is constructed from two masses, \( m = M \), little 'm' and capital 'M', which are connected by a massless string and supported by two frictionless pulleys; and all this takes place in a vacuum (in more ways than one), all very realistic and in accordance with the assumptions made in an introductory physics course. The bob, little 'm', is free to swing, like a simple pendulum, in the plane. The block, capital 'M', is confined to move in the vertical direction only, up and down, in one dimension. We further assume conservation of energy, there is no energy loss due to motion of the pulley, or bobs. Lastly, we choose to describe the problem in polar coordinates, where \( \Theta = \pi \) is measured from the plum line dropped from the pulley.

At this point we will quickly argue that the machine works when \( \Theta \) is greater than 90°, i.e., that there is always positive tension in the string: (go to board)

If we put the two pendula machine in the following configuration and let-go, what happens? Well, unfortunately, only the computer knows for sure, (pause, hopefully); but seriously, the string below little 'm' will not collapse. We recognize, along with Galileo, the constant acceleration of
all free falling bodies. Imagine that the string is no longer in the picture. Then \( m \) and \( M \) are free falling bodies with the same acceleration. When we place the string back into the picture, it is clear that the tension in the string is positive; it is zero when \( m \) is always upside down and directly above the pulley.

Before we continue, I would like to thank both Prof. Griffith and Prof. Crandall for generously extending their time and ideas about this problem. I am really beholden to both.

The first step is generally discovering what happens is to obtain the equations of motion. The standard operating procedure in these matters is to write down the energy equations and then to get the equations of motion by means of the Lagrange's equations. At this point one is usually in a position to give the equations a Newtonian interpretation, so that in the future, one can instantly derive the equations of motion by Newtonian principles.

To this end, we note that the kinetic energy for the system is:

\[
\mathbf{T} = \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m (r^2 \dot{\theta}^2) + \frac{1}{2} M \dot{r}^2
\]

\[
= \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + \frac{1}{2} M \dot{r}^2
\]

just like a simple pendulum but \( \dot{r} \neq 0 \).

And the potential energy is similar to that of an Atwood's machine with the addition of a \( \cos \theta \) term in order to complicate for the swinging side:

\[
\mathbf{V} = -mgr \cos \theta + Mgr + C
\]

depends on what we choose to zero the potential.
Recall that the Lagrangian is defined to be the difference between the kinetic and potential energies:

\[ \mathcal{L} = T - V = \frac{1}{2} M \dot{r}^2 + \frac{1}{2} m (\dot{r}^2 + (r \dot{\theta})^2) - Mgr + mgr \cos \theta - C \]

Use \( \mathcal{L} \) to calculate the equations of motion via Lagrange's equation:

\[ \frac{d}{dt} \left( \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) = \frac{\partial \mathcal{L}}{\partial q_i} \]

where \( q_1 = r \)

\( q_2 = \theta \)

So we proceed with the if we succeed in calculating the partial derivatives correctly we discover:

\[ r \text{-eq): } (m+M) \ddot{r} = mr \dot{\theta}^2 + mgr \cos \theta - Mg \]

\[ \theta \text{-eq): } \frac{d}{dt} (mr^2 \dot{\theta}) = -mgr \sin \theta \]

Should be familiar from the simple pendulum problem except \( r \neq 0 \).

As a notational convenience we define

\[ \alpha = \frac{M}{m} \]

and rewrite eq's of motion as

\[ r \text{-eq): } \left(1 + \alpha \right) \ddot{r} = r \dot{\theta}^2 + g (\cos \theta - \alpha) \]

\[ \theta \text{-eq): } \frac{d}{dt} (mr^2 \dot{\theta}) = -mgr \sin \theta \text{ or,} \]

\[ \ddot{\theta} + \frac{2\alpha}{r} \dot{\theta} + \frac{1}{2} g \sin \theta = 0 \]
My thesis is, in a sense, all on the board in this last box. It amounts to an analysis of these equations. On first inspection, these equations look horrible: the trigonometric terms and the squared terms make them 'horribly non-linear' (where horribly is used in the technical sense). But it is exactly the 'mystique' of non-linearity which boring me to study their properties. To my mind, they are not horrible, but a rather lovely set of second-order coupled non-linear ordinary differential equations. I want to break these coupled ordinary differential equations.

One really can't hope to find any exact solutions to these equations (there is a known one, but it is for negative mass, alas). The probability of finding such solutions is something like Lebesgue measure zero.

In order then to develop some feel for the system and "soften up the terrority" so to speak, we began our studies with extensive numerical integration studies which we later used as a yardstick to check our results.

Show SET 1 pictures:

What do you think you are, a king?

Commentary:

1) Describe initial conditions.
2) Mention each picture shows a mass increase of about 0.1.
3) For M<<1, it appears that \( r(t) \) presents runaway solutions, it is unbounded and never returns. (We will prove this)
4) The motion appears ergodic, periodic, bounded, stable (recurrent) etc. Voodoo

These pictures suggest we look into the following problems and introduce some terminology.
We would quickly like to present a simple and tentative scheme by which to classify the motions of the swinging bob in the plane.

\[ r(t) \leq r_{\text{max}} < \infty \]

\[ 0 < r_{\text{min}} \leq r(t) \leq r_{\text{max}} < \infty \]

\[ r(t + T_2) = r(t) \text{ and } \theta(t + T_2) = \theta(t) \]

where \( T_2 = nT_1 \) for some rational \( n \).

Then \( \theta(t + T_3) = \theta(t) \text{ if } r(t + T_3) = r(t) \).

If the bob always remains within some circle with radius \( r_{\text{max}} \), we shall say the motion is bounded; i.e., if there exists a positive upper bound for \( r(t) \).

If there further exists an \( r_{\text{min}} \) such that i.e., \( r \) has a non-zero lower bound, then we will call the motion stable.

Lastly, there appear to exist exactly periodic solutions.

The door now opens to a whole series of problems. Some of the more salient ones are:

1. Under what conditions do periodic orbits exist or can we find approximate or exact solutions for these trajectories?

2. Are the motions around periodic orbits stable (Poincaré orbital stability) in the sense that a solution near a periodic solution always stays close to it? Is the motion recurrent in Birkhoff’s sense?

3. Can we discover a classification scheme for the periodic orbits; or more specifically, analytic statements about under what conditions, bounded, stable and periodic motion will occur.

and so on.
I would like to show you a second set of trajectories, this time of orbits which are suspected to be periodic on numerical grounds and then to present some of the more custody results which have been obtained in trying to solve these and related questions.

Show: SET 2

Commentary:

1) There appears to be a ‘spectrum’ of periodic solutions which are even or odd in the sense that rotation by $180^\circ$ is even and $360^\circ$ is odd, symmetry with $y$-axis.
As a next step we examine a couple of limiting cases. We will look at:

4) Atwoods approximation: \( \theta(t) = 0, \forall t \).

2) Simple pendula approximation: \( \ddot{r} = 0 \), \( \theta \ll 1 \), \( \forall t \).

For the atwoods approximation we need only look at the \( r \)-eq to see that the name is aptly chosen:

\[(1 + \alpha) \ddot{r} = \dot{r}^2 \theta^1 + g(\cos \theta - \alpha)\]
\[= g(1 - \alpha) \Rightarrow \]
\[\ddot{r} = \frac{g(1 - \alpha)}{(1 + \alpha)}\]

which is the acceleration of an atwoods machine with solution:

\[r(t) = l + vt + \frac{1}{2} g \frac{(1 - \alpha)}{(1 + \alpha)} t^2. \quad \forall v = \dot{r}_0\]

The simple pendulum approximation turns out to be somewhat less trivial, the \( \theta \)-eq becomes:

\[\ddot{\theta} = -\frac{g}{l} \sin \theta \Rightarrow -\frac{g}{l} \theta \quad (\theta \ll 1).\]

The solution is a simple harmonic oscillator:

\[\theta(t) = \theta_0 \cos \omega t, \quad \omega = \sqrt{\frac{g}{l}}\]

Plugging this into the \( r \)-eq and examining the time average leads to a very interesting conclusion.
\[(1 + \alpha) \dot{r} = \langle r \dot{\theta}^2 \rangle + g \langle \cos \theta \rangle \dot{\theta} - g \alpha \], recall \( r = \ell \Rightarrow \dot{r} = \ell \dot{\theta} \)

\[g \alpha = \langle r \dot{\theta}^2 \rangle + g \langle \cos \theta \rangle \]

\[
\theta = \dot{\theta} \cos \omega t \\
\dot{\theta} = \dot{\theta} = \omega \sin \omega t \\
\theta = -\theta_0 \omega^2 \sin \omega t \quad \text{so} \\
\cos \omega t = 1 - \frac{\omega^2}{2} + \ldots \\
\text{and} \quad w^2 = \frac{g \ell}{\theta_0}
\]

\[g \alpha = \langle 1 \theta_0^2 \omega^2 \sin^2 \omega t \rangle + g \langle \cos (\theta_0 \cos \omega t) \rangle \\
\text{so} \quad \alpha \approx \frac{1}{2} \theta_0^2 + 1 - \frac{1}{4} \frac{\theta_0^2}{2}
\]

So

\[
\alpha = 1 + \frac{\theta_0^2}{4} \quad \text{STABILITY FORMULA}
\]

That is, if we can get a solution which resembles a simple pendulum for small angles then we must have the above relation holding for \( \alpha = \theta_0 \). Notice that this relation is independent of \( g \) and \( \ell \). Thus really has to be the case since both \( g \) and \( \ell \) scale in the original equations of motion, a non-trivial observation 7.

\( \alpha = f(\theta) \Rightarrow \) periodic solution

Such a relation we call a 'stability formula'. This formula agree well with numerical results for small angles \( (30^\circ < \) ). Others have been constructed by numerical methods for different types of periodic solutions.
\[ T = \frac{1}{2} (M+m) r^2 + \frac{1}{2} m r^2 \dot{\theta}^2 \quad V = g r (M - m \cos \theta) \]

\[ L = T - V \]

\[ L = \frac{1}{2} (M+m) r^2 + \frac{1}{2} m r^2 \dot{\theta}^2 + g r (m \cos \theta - M) \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i} \quad q_1 = r \quad q_2 = \theta \]

\[ \frac{2L}{2r} = m r \dot{\theta}^2 + g (m \cos \theta - M) \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial r} \right) = \frac{d}{dt} \left( (m+M) r \dot{r} \right) = (m+M) \ddot{r} \]

\[ (m+M) \ddot{r} = m r \dot{\theta}^2 + g (m \cos \theta - M) \]

\[ F_r = F_c + F_0 \]

\[ \frac{2L}{2\theta} = -g r m \sin \theta \]

\[ \frac{d}{dt} \frac{2L}{2\theta} = \left( m r^2 \dot{\theta} \right) = -g r m \sin \theta \]

\[ \frac{d}{dt} (L) = 0 \]

\[ 2r \dot{r} \dot{\theta} + r \ddot{\theta} = -g / \sin \theta \]

\[ r \ddot{\theta} + z \dot{\theta} = -g / \sin \theta \]

\[ \theta + z \dot{\theta} = -g / \sin \theta \]

\[ \text{angular equation} \]
Boundedness $\mu > 1$

$$E = T + V$$

$$E = \frac{1}{2} (1 + \mu) v^2 + r^2 \dot{\theta}^2 + gr(\mu - \cos \theta)$$

$E \geq gr(\mu - \cos \theta)$

$E \geq \frac{g r(\mu-1)}{r(\mu-\cos \theta)}$ always true

Now if $\mu > 1$ then $(\mu - \cos \theta) > 0$

Converse is also true

The $\leq$ also true had a little more tricky to prove.
Copied oscillators

Wilbefoe Pendulum
p. 128 French