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# NON-INTEGRABILITY OF THE DEGENERATE CASES OF THE SWINGING ATWOOD'S MACHINE USING HIGHER ORDER VARIATIONAL EQUATIONS

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Abstract. Non-integrability criteria, based on differential Galois theory and requiring the use of higher order variational equations ($\mathrm{VE}_k$), are applied to prove the non-integrability of the Swinging Atwood's Machine for values of the parameter which can not be decided using first order variational equations ($\mathrm{VE}_1$).

1. **Introduction and statement of results.** The Swinging Atwood's Machine (SAM for short) is a two-degrees-of-freedom Hamiltonian system derived from the well-known simple Atwood's machine. We refer to [13] and references therein for a derivation of the equations, even in the case that the effect of pulleys is considered. Historical and experimental results can be found in the same reference.

The Hamiltonian of the system is

$$H = \frac{1}{2}\left(\frac{p_r^2}{1+\mu} + \frac{p_\theta^2}{r^2}\right) + r(\mu - \cos\theta), \tag{1}$$

where $\mu$ is a mass ratio, $\mu > 1$ in the domain of interest. Other physical parameters have been normalised by taking suitable units.

We are interested on the integrability or non-integrability of (1). In general, we can consider a Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q},$$

where $H$ is assumed to be real analytic on some domain $\Omega$ of $\mathbb{R}^{2n}$. We consider the extension to a complex domain $\hat{\Omega}$ of $\mathbb{C}^{2n}$.

If $x = \{q, p\} \in \mathbb{C}^{2n}$ we consider solutions $x(t)$ with $t \in \hat{D} \subset \mathbb{C}$. The image of $\hat{D}$ by $x$ is a Riemann surface $\mathcal{R}$.

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We shall consider integrability in the Liouville-Arnol'd sense:

**Definition 1.** A Hamiltonian system is integrable if and only if there exist $n$ first integrals $f_1, f_2, \dots, f_n$ independent almost everywhere and in involution. Usually it is taken $f_1 = H$. In general the functions $f_1, f_2, \dots, f_n$ will be considered meromorphic in a neighbourhood of a given solution $x(t)$.

The standing problem is to find *necessary conditions for integrability*, or, equivalently, *sufficient conditions for non-integrability*.

Integrable Hamiltonian systems have, in some sense, well ordered dynamics, while non-integrable ones are associated to some amount of *chaos*. It may be the case that chaotic dynamics appears only in the non-real part of the complex phase space without showing up in the real one (see, e.g. [11]). A chaotic behaviour implies lack of predictability, i.e., a sensitive dependence to initial conditions.

Several criteria follow from the so-called Morales-Ramis theory, which includes classical results by Ziglin [16]. The results summarized here are contained in [8, 9]. See also [7] for all the necessary background and technical details.

Consider the $m$-dimensional ODE $\dot{x} = f(x)$ and let $x(t)$ be a solution. The first variational equations (VE$_1$) along $x(t)$ are given by $\frac{d}{dt}A = Df(x(t))A$ and we consider the initial condition $A(t_0) = Id$, where $x_0 = x(t_0)$ is a regular point of $f$ and $Df$. If we take closed paths on the Riemann surface $\mathcal{R}$ with base point $x_0$, one can associate to each path the corresponding monodromy matrix, that is the matrix $A$ at the end of the path. The set of all these matrices form the monodromy group.

More generally, we can consider any linear ODE

$$\frac{d}{dt}A(t) = B(t)A(t). \tag{2}$$

We assume that the entries of $B$ belong to some field of functions $K$. Let $\xi_{i,j}$ be the elements of a fundamental matrix of (2) and $L$ be the *extension* $K(\xi_{1,1}, \xi_{1,2}, \dots, \xi_{m,m})$, which is trivially a differential field. Consider the *Galois group* $G = \text{Gal}(L \mid K)$, that is the group of automorphisms of $L$ leaving the basic field $K$ invariant. It is an algebraic group. Then the following result is obtained.

**Theorem 1.** *(Morales-Ramis) Under the assumptions above if a Hamiltonian is integrable in a neighbourhood of $\mathcal{R}$ then the identity component $G^0$ of the Galois group of the first order variational equations $VE_1$ along $\mathcal{R}$ is commutative.*

The identity component is taken using the Zariski topology. We also recall that if the singular points of (2) are of singular regular type, then the Galois group coincides with the Zariski closure of the monodromy group (a result sometimes called Schlesinger's theorem [14], and which is a special case of the Ramis density theorem [4]). This happens in present case, the first variational equations (11) being a simple case of hypergeometric equation.

A delicate example of application of Theorem 1 can be seen in [12]. See also [10] for a long, but not exhaustive, list of examples where this theorem has been used to detect non-integrability.

Concerning SAM problem the following result was proved in [3] using Ziglin's theory

**Theorem 2.** *The Hamiltonian system defined by (1) is non-integrable if $\mu \neq \mu_p$ where $\mu_p = 1 + \dfrac{4}{p^2 + p - 4}$, $p \in \mathbb{N}$, $p \geq 2$.*

Furthermore the case $p = 2$, $\mu_p = 3$ is known to be integrable [15].

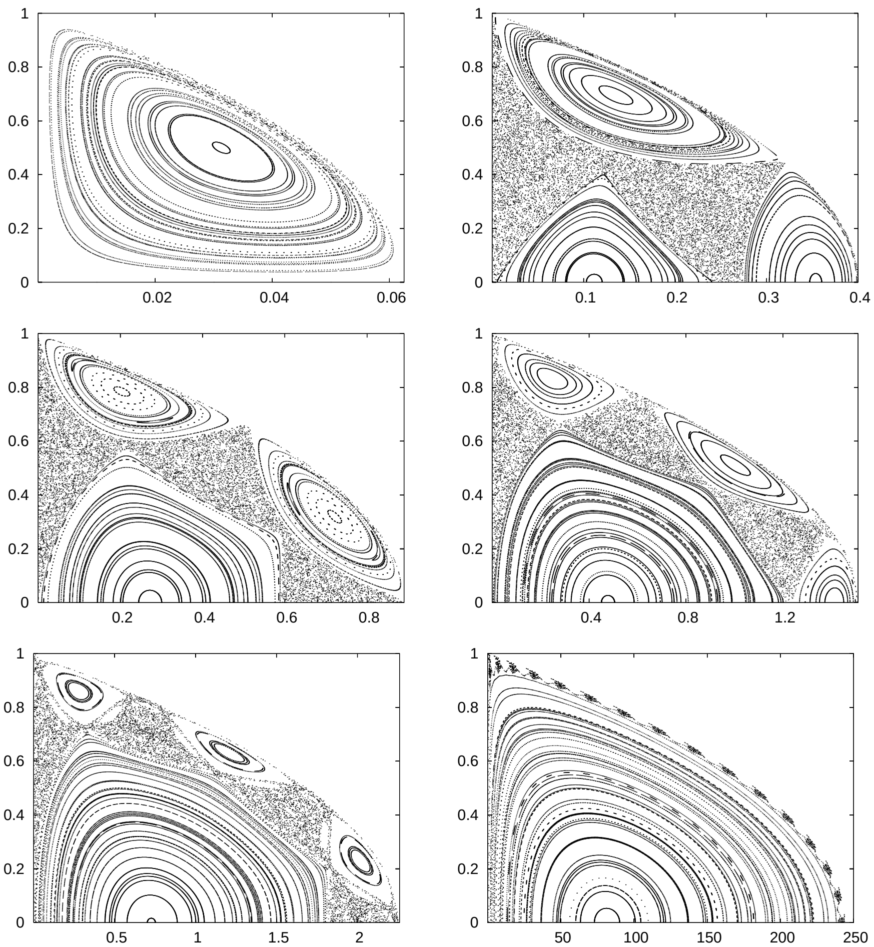

FIGURE 1. Poincaré sections of (1) through $\theta \equiv 0 \pmod{2\pi}$, $p_\theta > 0$ on the energy level $H = 1/(2(1+\mu))$. Due to the symmetry only the upper part is shown in $(r, p_r)$. From left to right and top to bottom $\mu = \mu_p$ for $p = 2, 3, 4, 5, 6, 62$ are shown.

In the "degenerate" cases $\mu = \mu_p$, $p > 2$ the variational equations $\mathrm{VE}_1$ give nothing against integrability using as solution $x(t)$ the very simple solution (5). We shall argue in what follows why it is quite natural to take this solution to try to prove non-integrability.

Note that the value of $\mu_p$ tends to 1 as $p \to \infty$. On the other hand, for these exceptional cases a Poincaré section reveals that the system is far from integrable (see Figure 1). For $\mu_2 = 3$ the integrable structure is clearly seen. Other values, like $\mu_3 = 3/2$, $\mu_4 = 5/4, \mu_5 = 15/13, \mu_6 = 21/19$, display large chaotic zones. However, when $\mu_p$ is close to 1, as happens for $p$ large, the only hint on non-integrability comes from the presence of tiny chains of islands. For instance, for $p = 62, \mu_p = 1953/1951$ additional explorations, see Figure 2, show the existence of chains of islands of periods 31, 32 and 62 very close to the boundary of the domain (compared to the size of the domain). In all cases one has taken a level of energy $H = 1/(2(1+\mu))$ so that an orbit on the invariant plane $\theta = p_\theta = 0$ passing through $(r, p_r) = (0, 1)$ is at the boundary of the domain of definition of the Poincaré map. This will be the solution $x(t)$ used to prove the non-integrability. The dynamic idea beyond this choice is the numerical evidence that for large $p$ the observed chaotic behavior in the Poincaré section is close to that boundary.

To produce the plots in Figure 1 one has taken a few initial points on a grid in $(r, p_r)$ and 1000 Poincaré iterates have been computed from each one of them.

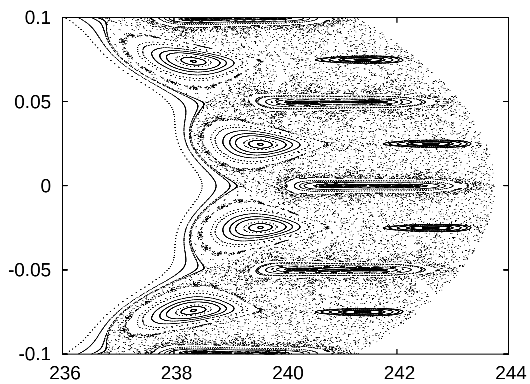


FIGURE 2. Magnification of a small domain of Figure 1, using more initial points, to show the existence of islands of periods 31,32 and 62.

The fact that $G^0$ is commutative for $\mu = \mu_p, p > 2$ when $x(t)$ is taken as given by (5) and, hence, there is nothing against integrability, suggests to try to detect non-integrability at *higher order*. The theoretical support is given as follows (see [5]).

Let $\varphi(t, x_0)$ be the solution of $\dot{x} = f(x(t))$ with $\varphi(t_0, x_0) = x_0$. We consider as *fundamental* solutions of the $k$-th order variational equations, $\mathrm{VE}_k$, based on $x_0$, the string of maps $(\varphi^{(1)}(t), \varphi^{(2)}(t), \ldots, \varphi^{(k)}(t))$ such that

$$\varphi(t, y_0) = \varphi(t, x_0) + \varphi^{(1)}(t)(y_0 - x_0) + \ldots + \varphi^{(k)}(t)(y_0 - x_0)^k + \ldots,$$

for every $y_0$ sufficiently close to $x_0$. Obviously $\varphi^{(1)}(t)$ is a solution of the first order VE=VE$_1$ with the initial condition $\varphi^{(1)}(t_0) = Id$. If we write $y_0 = x_0 + \xi$ then the Taylor expansion of $\varphi(t, x_0 + \xi)$ has $\varphi^{(k)}(t)(\xi)^k$ as homogeneous part of order $k$, that is the above string of maps contain all the entries of the $k$-jet.

The $\varphi^{(k)}(t)$ satisfy linear non-homogeneous ODE, the non-homogeneous part depending in a nonlinear way of the entries of the previous maps $\varphi^{(1)}(t), \ldots, \varphi^{(k-1)}(t)$. If we consider *simultaneously* the differential equations for $\varphi^{(1)}(t)$ to $\varphi^{(k)}(t)$ the system is nonlinear but, for any $k$, the equations for the entries of the $\varphi^{(j)}, j = 1, \ldots, k$ can be made linear by introducing additional variables (products of entries) which also satisfy linear ODE (see [10] for details). The initial conditions are

$$\varphi^{(1)} = Id, \qquad \varphi^{(k)}(t_0) = 0 \quad \text{for} \quad k > 1. \tag{3}$$

See [10] for explicit versions in terms of the entries of the maps $\varphi^{(k)}(t)$. For further use we introduce the notation $x_i$, $x_{i;k}$, $x_{i;k_1,k_2}$, $x_{i;k_1,k_2,k_3}, \ldots$ for the entries of $x$ and the first, second, third, ... derivatives with respect to the initial conditions. Divided by the corresponding factorial they give the entries of

$\varphi(t), \varphi^{(1)}(t), \varphi^{(2)}(t), \varphi^{(3)}(t), \ldots$. For instance $x_{4;1,3,3} = \frac{\partial^3(\varphi)_4}{\partial x_1 \partial x_3^2}$. Note that once $\varphi^{(1)}$ is available, all the entries of $\varphi^{(k)}, k = 2, 3, \ldots$ are obtained by quadratures.

Hence, one can introduce the *k-th order Galois group* $G_k$ as the Galois group associated to the linearized version of the variational equations up to order $k$. We can also introduce the $k$-th order monodromy as the monodromy obtained with the linearized version of the $\mathrm{VE}_k$. The information it gives is equivalent to the information obtained by transporting the jet up to order $k$. That is, starting at the point $x_0 + \xi$ at time $t_0$ one has

$$\varphi(t; t_0, x_0 + \xi) = \sum_{0 \le |j| \le k} a_j(t) \xi^j + \mathcal{O}(|\xi|^{k+1}),$$

where $j$ is a multiindex and the $a_j$ coefficients are $m$-dimensional vectors if $x$ is $m$-dimensional. If later we are interested in $\varphi(s; t_0, x_0 + \xi)$ when we move from $t$ to $s$ along a given path (note that one can have $s = t$) we have to "transport the jet" from $t$ to $s$, which justifies naming this as jet transport.

The jet $\sum_{0 \le |j| \le k} a_j(t_1) \xi^j$ when we return to $x_0$ moving along a closed path $\gamma$ from $t_0$ to $t_1$ with $\gamma(t_0) = \gamma(t_1) = x_0$, can be seen as the $k$-th order monodromy along $\gamma$ with base point $x_0$, to be denoted as $M_k^\gamma$. In our present problem, however, all the paths that we shall consider have the same value for $t_0$ and $t_1$. The composition of jets like $M_k^\gamma$ using different paths $\gamma$ forms a group, to be denoted simply as $M_k$, which is a natural extension of the monodromy group $M_1$.

Then, for any $k \ge 1$ the following extension of Theorem 1 holds:

**Theorem 3.** *([10]) Under the assumptions above if the Hamiltonian is integrable in a neighbourhood of $\mathcal{R}$ then for any $k \ge 1$ the identity component $(G_k)^0$ of $G_k$ is commutative.*

This result gives rise to *non-integrability criteria* to all orders. Note that these criteria can depend strongly on the reference solution $x(t)$ and on the paths $\gamma$ used to transport the jet. In general it is not true that if these necessary criteria are satisfied for all $k \in \mathbb{N}$ the system is integrable. The problem of finding sufficient conditions for integrability remains open.

The main purpose of that paper is to use Theorem 3 to prove

**Theorem 4.** *The degenerated cases $\mu = \mu_p$, $p > 2$ of the SAM are non-integrable.*

The result will follow from the non-commutativity of $(G_3)^0$ that it is proved using suitable paths along a solution on the invariant plane $\theta = p_\theta = 0$.

As explained in [5] the first step will be to take two closed paths, that in present case are denoted as $\gamma_+$ and $\gamma_-$, such that the monodromies $M_1^{\gamma_\pm}$ are in $(G_1)^0$. As it will be seen in the proof of Theorem 4 the $\mathrm{VE}_1$ on the plane $\theta = p_\theta = 0$ decouple in the $(r, p_r)$ and the $(\theta, p_\theta)$ variables. As the subproblem in $(r, p_r)$ variables is integrable one should only take care of $\mathrm{VE}_1$ in the $(\theta, p_\theta)$ variables. The fact that the $M_1^{\gamma_+}$ and $M_1^{\gamma_-}$ are in $(G_1)^0$ follows immediately from the unipotent character of these matrices. Then one has also that $M_k^{\gamma_+}$ and $M_k^{\gamma_-}$ are in $(G_k)^0$ (eventually one has to replace the Riemann surface $\mathcal{R}$ by a "subsurface" $\mathcal{R}'$) and the lack of commutativity for $k = 3$ is enough to prove Theorem 4. See Lemma 2 in [5] for additional details.

We can interpret that result in terms of jet transport. After transporting along $\Gamma = \gamma_-^{-1} \circ \gamma_+^{-1} \circ \gamma_- \circ \gamma_+$ the initial variations $\xi$ we recover, $\xi$ at first order, zero at second order and something different from zero at third order. In fact, we do not claim that the second order terms are zero, despite we have a strong evidence by explicit symbolic computation for low values of $p$ (up to several thousands). But there are definitely, third order terms different from zero.

Additional examples on the use of higher order variational equations to detect non-integrability and a methodology to deal with these problems can be found in [5].

A big effort has been undertaken to compute the monodromy for many linear differential equations, but the authors are not aware of a similar effort concerning *higher order monodromy*, i.e., the study of the transport of jets of arbitrary order.

In general no explicit solution is known for an arbitrary Hamiltonian. But assume we are able to find, numerically, two paths $\psi_1, \psi_2$, such that $M_1^{\psi_j}$ are in $(G_1)^0$, and we can compute $M_k^{\psi_j}, j = 1, 2$ along them. Then

$$[M_k^{\psi_1}, M_k^{\psi_2}] = (M_k^{\psi_2})^{-1} \circ (M_k^{\psi_1})^{-1} \circ M_k^{\psi_2} \circ M_k^{\psi_1}, \tag{4}$$

should be trivial, that is, equal to the identity to order $k$ if the system is integrable. If it does not hold and we can rigorously prove that this is still true when we account for the numerical errors, then non-integrability is proved.

Furthermore the present case involves in an essential way two different singularities. This means that it requires some *global* information in contrast with other problems in which the lack of integrability can be detected by *local* computations. See, e.g., the degenerate Hénon-Heiles system used in [10] as simple example.

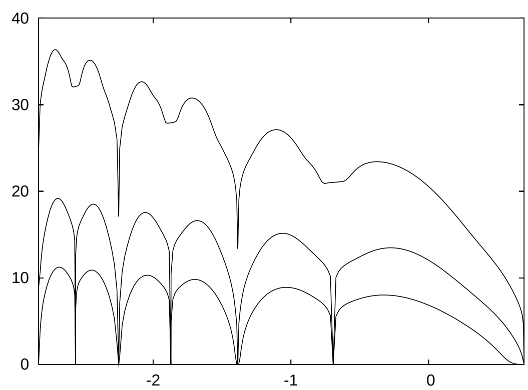


FIGURE 3. Norms $g^{(k)}$: The lower (resp. upper) curve corresponds to $k = 1$ (resp. $k = 3$). As $\mu_p$ accumulate to 1 like $1 + \mathcal{O}(p^{-2})$ the horizontal axis displays $\log(\mu - 1)$ while in the vertical one $\text{argsinh}(g^{(k)})$ is shown.

A systematic approach to check numerically for non-integrability in an efficient way, based on Theorem 3, illustrations concerning the SAM and a variety of additional examples can be found in [6]. This numerical information has been very useful to suggest the approach to be taken for the proof of Theorem 4. As an illustration Figure 3 displays the values of the norms $g^{(k)}$ of the terms of order $k$ of the jet after the transport along $\Gamma$ for $k = 1, 2, 3$ as a function of $\mu$. As norm we have taken $g^{(k)} = \sum_{|j|=k} |(a_j)^f|$ where $(a_j)^f$ denotes the value of $a_j(t)$ at the end of the full path. For the order $k = 1$ one has subtracted the identity from $\varphi^{(1)}$. The concrete path used for the computations will be shown in Figure 4.

The places where $g^{(1)}$ and $g^{(2)}$ can be considered equal to zero correspond, from right to left, to $p = 2\,(\mu = 3), p = 3\,(\mu = 3/2), \ldots, p = 8\,(\mu = 18/17)$. For $p = 2$ also $g^{(3)}$ is seen, numerically, equal to zero, in agreement with the theoretical fact that the case $p = 2$ is integrable. For all other values of $\mu$ one already has numerical evidence that $g^{(1)} \neq 0$. A simple estimate of errors in the values that are suspected to be zero at the end of the path is obtained as follows: the modulus of the coefficient after integrating the ODE along $\Gamma$ is divided by the maximum value reached during the integration. Concretely, if we denote as $a^*(t)$ one of these coefficients, we compute $|(a^*)^f|/\max_{t\in\Gamma}|a^*(t)|$. This estimate is reasonable because of the possible cancellations if $\max_{t\in\Gamma}|a^*(t)|$ is large. In all cases if computations are done with $d$ binary digits that quotient is below $100 \times 2^{-d}$. Typical ODE integrations are done with Taylor method of order 26 and local truncation error $10^{-20}$ when working in double precision and with order 45 and local truncation error $10^{-35}$ when quadruple precision is used.

2. **Sketch of the proof of Theorem 4 and first steps.** Guided by numerical results (see [6]) we confine our theoretical study to third order variational equations. It will be proved that this is enough to detect non-integrability.

We shall use the following notation. If $\gamma, \psi$ are two closed paths on a Riemann surface $\mathcal{R}$ then $\psi \circ \gamma$ will denote the path obtained by following first $\gamma$ and then $\psi$. Similar for a larger number of paths. A path traveled in reversed direction will be denoted as $\gamma^{-1}$. Furthermore we shall also use the same notation, say $\gamma$, for a closed path on a Riemann surface $\mathcal{R}$ lying on the (complex) phase space and for the corresponding temporal arc in the domain of definition $\hat{D}$ of $x(t)$. The meaning will be clear from the context.

The proof proceeds in several steps:

- Selection of a simple, regular, solution to (1) such that the variational equations along it have two singularities.
- Second step is the selection of a suitable path $\Gamma$, which is obtained from the composition of simple paths $\gamma_+$ and $\gamma_-$ around the singularities. More concretely, we shall take $\Gamma = \gamma_-^{-1} \circ \gamma_+^{-1} \circ \gamma_- \circ \gamma_+$. Then the commutator $(M_k^{\gamma_-})^{-1} \circ (M_k^{\gamma_+})^{-1} \circ M_k^{\gamma_-} \circ M_k^{\gamma_+}$, of the form (4), is simply represented as $M_k^{\Gamma}$.

  This is a key point because other choices can lead to more involved computations.

  One checks that $M_1^{\gamma_+}$ and $M_1^{\gamma_-}$ are in $(G_1)^0$. From this it follows that $M_k^{\gamma_+}$ and $M_k^{\gamma_-}$ are in $(G_k)^0$ for a suitable Riemann surface, see [5].
- The solutions of the variational equations for the different orders (equivalent to the coefficients of the jet) satisfy symmetry relations as a function of $t$ and some of them are identically zero. The transport of the third order jet along $\Gamma$, $M_3^{\Gamma}$, can be expressed from the coefficients of the transport of the jet along $\gamma_+$ and several additional integrals. For the computation of integrals along paths in complex time one has to take into account that, if the paths start, say, at $t = 0$ they can return to the same value of $t$ with a different determination of the function to be integrated. This is examined in detail.
- At that point we claim that some of the coefficients in $M_3^{\gamma_+}$ are zero and some are different from zero. Then a part of $M_3^{\Gamma}$ can be computed and this is enough to prove Theorem 4.

Let us write the Hamiltonian vector field for (1) in the form $\dot{x} = f(x)$ and let $(x_1, x_2, x_3, x_4) = (r, \theta, p_r, p_\theta)$ and $f_i$, $i = 1, \ldots, 4$ be the entries of $f$.

A simple, regular, solution to (1) on the invariant plane $x_2 = x_4 = 0$, is given by

$$x_1(t) = r(t) = \frac{1}{a}\left(1 - t^2\right), \quad x_2(t) = \theta(t) = 0, \quad x_3(t) = p_r(t) = (1-\mu)t, \quad x_4(t) = p_\theta(t) = 0, \tag{5}$$

where $a = p^2 + p - 2$ and from now on we shall use simply $\mu$ instead of $\mu_p$, but keeping in mind that only the values corresponding to integer $p$ are considered. Note that $r(\pm 1) = 0$. The solution (5) is somewhat arbitrary, because the initial value of the radius $x_1(0)$, assuming $x_3(0) = 0$, can be any positive number. This depends on the level of energy in which the solution is considered. Using (5) the level of energy is $\frac{4}{(p^2+p-2)(p^2+p-4)}$ which is different from the value used in Figures 1 and 2. But if we scale $x_1(0)$ by $\nu^2$ then $r(\pm\nu) = 0$ and the level of energy changes. The derivatives of variable $i$ of orders $(j_1, j_2, j_3, j_4)$ with respect to $(x_1, x_2, x_3, x_4)$ scale like $\nu^{n(i)-2j_1-j_3-3j_4}$, where $n(1) = 2, n(2) = 0, n(3) = 1, n(4) = 3$. Hence, scaling keeps the commutativity properties of $M_k$. The effect of the scaling will be seen also in [6], where it is used to enhance or decrease the numerical difficulties, by decreasing or increasing, respectively, the distance of the path to the singularities.

The solutions of the first variational equations associated to the variables $(x_1, x_3)$ are also elementary

$$\begin{pmatrix} x_{1;1}(t) & x_{1;3}(t) \\ x_{3;1}(t) & x_{3;3}(t) \end{pmatrix} = \begin{pmatrix} 1 & t/(1+\mu) \\ 0 & 1 \end{pmatrix}. \tag{6}$$

On the other hand, the first variational equations associated to the $(x_2, x_4)$ variables are

$$\frac{d}{dt}\begin{pmatrix} x_{2;2}(t) & x_{2;4}(t) \\ x_{4;2}(t) & x_{4;4}(t) \end{pmatrix} = \begin{pmatrix} 0 & r^{-2}(t) \\ -r(t) & 0 \end{pmatrix}\begin{pmatrix} x_{2;2}(t) & x_{2;4}(t) \\ x_{4;2}(t) & x_{4;4}(t) \end{pmatrix}. \tag{7}$$

All the other entries of the $\mathrm{VE}_1$ are identically zero.

Typically we shall use a notation like $x_{i;k_1}(t), x_{i;k_1,k_2}(t), \ldots$ to denote the functions as depending on $t$, while $x_{i;k_1}, x_{i;k_1,k_2}, \ldots$ will denote the values at the end of a path which will be clear from the context.

While (5) is not introducing any singularity, (7) does at $r = 0$. Note that the solution through $r = 0$ is non-physical. But this is irrelevant for the proof on the non-integrability.

This fact suggests to take the following paths: Let $\gamma_+$ (resp. $\gamma_-$) be a closed path starting, in the temporal domain, at $t = 0$ and going around $t_+ = 1$ (resp. $t_- = -1$) clockwise. It is convenient to take each of the paths symmetrical with respect to the real axis. The full path will be $\Gamma = \gamma_-^{-1} \circ \gamma_+^{-1} \circ \gamma_- \circ \gamma_+$, as mentioned. The initial conditions are taken from (5) with $t = 0$. The symmetries associated to the four paths involved in $\Gamma$ will play a relevant role, but other parts of the proof require an explicit knowledge of the transport of the jet to third order along $\gamma_+$.

Figure 4 sketches a possible model for the paths $\gamma_\pm$ and the complete path $\Gamma$.

It is clear that there is freedom in the definition of the basic paths $\gamma_+$ and $\gamma_-$. One could take one of them clockwise and the other counterclockwise. But with present definition we have that $\gamma_-$ is obtained by changing the sign of $\gamma_+$. Furthermore the path $\gamma_-^{-1} \circ \gamma_+^{-1}$ is the complex conjugate of $\gamma_- \circ \gamma_+$. As a consequence it will be seen that one can recover all the necessary information from the transport along $\gamma_+$.

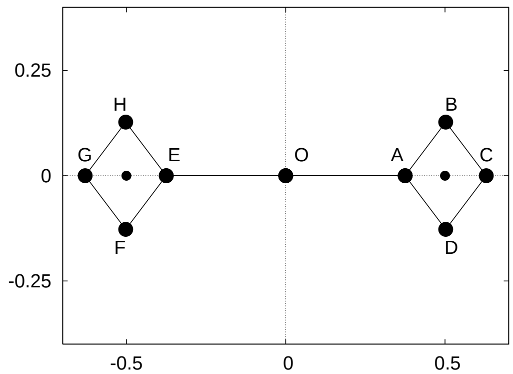


FIGURE 4. Sketch of the path to be used as model for $\Gamma$. In the time domain $\Gamma$ is OABCDAOEFGHEOADCBAOEHGFEO. The paths $\gamma_+, \gamma_-$ or OABCDAO, OEFGHEO, are traveled clockwise. Note that each path is symmetrical with respect to the real axis.

For the proof of Theorem 4 we shall show that $\mathrm{VE}_1$ gives the identity along $\Gamma$, then $\mathrm{VE}_2$ is zero (with one eventual exception, see Remark 1) and some of the elements in $\mathrm{VE}_3$ are different from zero.

Using the solution (5) one can compute the coefficients of the variational equations which are different from zero along it:

$$\begin{array}{llll} f_{1;3} = (1+\mu)^{-1}, & f_{2;4} = x_1^{-2}, & f_{4;2} = -x_1, & \\ f_{2;1,4} = -2x_1^{-3}, & f_{3;2,2} = -1, & f_{3;4,4} = 2x_1^{-3}, & f_{4;1,2} = -1, \\ f_{2;1,1,4} = 6x_1^{-4}, & f_{3;1,4,4} = -6x_1^{-4}, & f_{4;2,2,2} = x_1, & \end{array} \tag{8}$$

where $f_{i,k_1}, f_{i,k_1,k_2}, f_{i,k_1,k_2,k_3}$ denote derivatives of $f_i$ in the obvious way and we have not written the symmetric terms. All the functions in (8) are even in $t$.

Next two lemmas follow easily from inspection of the equations, their symmetries (locally, around $t = 0$), the variational equations, the form of the coefficients (8) and the initial conditions (3).

**Lemma 1.** *The parity of an element $x_i(t)$, $x_{i;k_1}(t)$, $x_{i;k_1,k_2}(t), x_{i;k_1,k_2,k_3}(t), \ldots,$ if it is not identically zero, is the same as the parity of*

$$\mathcal{P} = \#\{i, k_1, \ldots, k_s \in \{3, 4\}\},$$

*where $s$ denotes the order of the variationals. This fact holds for all order $s$.*

**Lemma 2.** *The elements of the form $x_{i;k_1,k_2}(t), x_{i;k_1,k_2,k_3}(t), \ldots$ which are not identically zero satisfy the following condition: The cardinality of the set $\{k_j \in \{2, 4\}\}$ must be non-zero and to have parity different from the parity of $i$. This fact holds for all order $s$.*

As mentioned the rule applies also to higher order derivatives. This gives, for instance, that from the total of 140 elements in the jets to order 3 of the four image variables (including order 0), only 55 are not identically zero. For large order, simple combinatorial computations show that the fractions of identically zero and non-identically zero elements tend to be the same. All this holds also for $p \in \mathbb{R}$, $p > p_m = (\sqrt{17} - 1)/2$, that is, for all values of $p$ such that $\mu_p > 0$.

Before dealing with statements about some coefficients being zero or non-zero at the end of $\gamma_+$ we should discuss the effect of the parity of $p$. The following proposition will be proved in Section 3.

**Proposition 1.** *Let $\Phi_2(t)$ be the solution of (7) which is the identity at $t=0$, that is, the fundamental solution. Then at the end of the path $\gamma_+$ it has the form*

$$\begin{pmatrix} 1 & x_{2;4} \\ 0 & 1 \end{pmatrix} \quad \text{for} \quad p \quad \text{odd and} \quad \begin{pmatrix} 1 & 0 \\ x_{4;2} & 1 \end{pmatrix} \quad \text{for} \quad p \quad \text{even,}$$

*where the respective coefficients $x_{2;4}, x_{4;2}$ are non-zero and purely imaginary. $M_1^{\gamma_+}$, $M_1^{\gamma_-}$ belong to $(G_1)^0$. Moreover, at the end of $\gamma_- \circ \gamma_+$, $\Phi_2$ becomes the identity.*

From now on, **we shall concentrate on the case $p$ odd**, the proofs being the same for the case $p$ even, taking into account that symmetry.

After Lemma 1, the only second order variables not identically zero are the following ones

$$x_{i;2,2}(t), \quad x_{i;2,4}(t), \quad x_{i;4,4}(t), \qquad\qquad i=1,3, \tag{9}$$

$$x_{j;1,2}(t), \quad x_{j;1,4}(t), \quad x_{j;2,3}(t), \quad x_{j;3,4}(t), \qquad j=2,4. \tag{10}$$

**Proposition 2.** *Assume $p$ is odd. The following coefficients are zero after going along $\Gamma$:*

$$x_{i;2,2}, x_{i;2,4}, \ \text{for } i=1,3, \quad x_{j;1,2}, x_{j;1,4}, x_{j;2,3}, \ \text{for } j=2,4, \quad \text{and also } x_{3;4,4}, x_{4;3,4}.$$

The proof of Proposition 2 will be given in Section 4.

**Remark 1.** Lemma 2 and Proposition 2 prove that all the elements of the second order variationals along $\Gamma$ are zero, except $x_{1;4,4}$ and $x_{2;3,4}$. If some of these elements is different from zero Theorem 4 would be proved. However, there is a strong numerical evidence that they are also zero at the end of $\Gamma$ (see the end of Section 1 and [6]). The proof of that is rather cumbersome as a look at the equations in (17), the expression of the function $\mathcal{D}_4(t)$ which involves the squares of $x_{2;4}(t)$ and $\frac{d}{dt}x_{2;4}(t)$, a function given in (13), allows to check. So we prefer to concentrate on suitable third order variationals whose analytical computation is easier and is independent of the fact that $x_{1;4,4}$ and $x_{2;3,4}$ are zero or non-zero. One should also mention that some of the relations in Proposition 2 follow from the symplectic character of the jet transport.

**Remark 2.** An alternative and essentially equivalent approach for the proof of Theorem 4 can be the computation of the transport of the jet to order 3 (or of a sufficient part of it) along $\gamma_+, \gamma_-, \gamma_+^{-1}$ and $\gamma_-^{-1}$ by using the symmetries which relate the jet transported along $\gamma_+$ to the other ones. Then the transport of the jet along $\Gamma$ is obtained by composition.

Next proposition ends the proof of Theorem 4.

**Proposition 3.** *Assume $p$ is odd. After the transport along $\Gamma$ the coefficients $x_{2;2,2,4}$ and $x_{4;2,4,4}$ are real and non-zero.*

The proof of Proposition 3 will be given in Section 4.

**Remark 3.** Numerical evidence that several other coefficients of the third order jet are zero at the end of $\gamma_+$ is reported in [6]. In fact, the only coefficients which are not zero after the transport along $\Gamma$, beyond the identity at order 1, seem to be $x_{2;2,2,4} = -x_{4;2,4,4}$ and $x_{2;4,4,4}$ in the case $p$ odd and, symmetrically, $x_{4;2,2,2}$ and $x_{2;2,2,4} = -x_{4;2,4,4}$ in the case $p$ even, except in the integrable case $p = 2$. But none of these evidences will be used in the proof.

3. **Study of first order variational equations.** The first thing we need is the solution of (7). Let us write as $(\xi, \eta)$ the entries of a column of the solution to (7). The system $\dot{\xi}(t) = r^{-2}(t)\eta(t)$, $\dot{\eta}(t) = -r(t)\xi(t)$ becomes

$$(1-t^2)\ddot{\xi}(t) - 4t\dot{\xi}(t) + a\xi(t) = 0, \tag{11}$$

recalling $a = p^2 + p - 2$. The singularities at $t = \pm 1$ are clear from (11). From a solution $\xi(t)$ we obtain $\eta(t) = r^2(t)\dot{\xi}(t)$. Equation (11) is a special case of the hypergeometric equation with integer parameters.

We look for two fundamental solutions of (11) $\xi_1(t), \xi_2(t)$. Except by scaling factors, to have the identity matrix at $t = 0$, they can be selected as follows:

- $\xi_1(t)$ is a polynomial of degree $p-1$, even if $p$ is odd and odd if $p$ is even. It is normalized in such a way that $\xi_1(1) = 1$. Then using (11) it satisfies that $\dot{\xi}(1) = a/4$. Except by a scaling factor it coincides with the Jacobi polynomial $P^{(1,1)}_{p-1}(t)$ (see, e.g. [1]), that is, it is proportional to $\dfrac{1}{1-t^2}\dfrac{d^{p-1}}{dt^{p-1}}\left((1-t^2)^p\right)$. With this normalization the expansion around $t = 0$ is of the form

$$\begin{array}{l} \dfrac{(-1)^{(p-1)/2}}{2^{p-1}} \begin{pmatrix} p-1 \\ (p-1)/2 \end{pmatrix} \dfrac{2}{p+1} + \mathcal{O}(t^2) \text{ for } p \text{ odd}, \\ \dfrac{(-1)^{(p-2)/2}}{2^{p-1}} \begin{pmatrix} p \\ p/2 \end{pmatrix} t + \mathcal{O}(t^3) \text{ for } p \text{ even}. \end{array} \tag{12}$$

  Using Stirling's formula (and taking into account the error!) the absolute value of the leading coefficients can be bounded from above by $\sqrt{\frac{8}{(p-1)\pi}}/(p+1)$ and $\sqrt{\frac{8}{p\pi}}$, respectively.
- $\xi_2(t)$ contains singularities and it is of the form

$$\xi_2(t) = \left[-\frac{1}{2}(a+2)\left(\log(1+t) - \log(1-t)\right)\xi_1(t) + \psi(t)\right] + g(t), \tag{13}$$

  where $\psi(t) = -\dfrac{2s(t)}{1-t^2}$, being $s(t) = 1$ for $p$ even, $s(t) = t$ for $p$ odd. Furthermore $g(t)$ is the unique polynomial solution of degree $p-2$ of the equation

$$(1-t^2)\frac{d^2g}{dt^2}(t) - 4t\frac{dg}{dt}(t) + ag(t) = (2a+4)\left(\frac{d\xi_1(t)}{dt} - \frac{t\xi_1(t) - s(t)}{1-t^2}\right). \tag{14}$$

  In fact, it is immediate to check that $\xi_2(t)$ is a solution of (7) if and only if, $g(t)$ is a solution of (14). Using the normalization $\xi_1(1) = 1$ and the parity of $\xi_1(t)$ one has that $t\xi_1(t) - s(t) = (1-t^2)Q(t)$ for some polynomial $Q(t)$ of degree $p-2$ which has the same parity as $p$. Then, a unique polynomial solution of (14) can be determined.
- A fundamental matrix is obtained by taking

$$\begin{pmatrix} \xi_1(t) & \xi_2(t) \\ \eta_1(t) & \eta_2(t) \end{pmatrix} \text{ if } p \text{ odd}, \quad \begin{pmatrix} \xi_2(t) & \xi_1(t) \\ \eta_2(t) & \eta_1(t) \end{pmatrix} \text{ if } p \text{ even}. \tag{15}$$

  We note that due to the normalization used for $\xi_1$, as shown in (12), the matrices above become diagonal at $t = 0$. Therefore to pass from (15) to the usual normalization for the fundamental matrix, that is, the identity at $t = 0$, one has to multiply (15) by some constant diagonal matrix, $C$, different for $p$ odd and $p$ even (this is irrelevant for the proofs, but affects the numerical computations as shown in [6]). In particular, if $p$ is odd we obtain

$$\begin{pmatrix} x_{2;2}(t) & x_{2;4}(t) \\ x_{4;2}(t) & x_{4;4}(t) \end{pmatrix} = \begin{pmatrix} c_1\,\xi_1(t) & c_2\,\xi_2(t) \\ c_1\,\eta_1(t) & c_2\,\eta_2(t) \end{pmatrix} \tag{16}$$

for some constants $c_1, c_2$.

**Proof of Proposition 1**. Assume $p$ is odd. One has to check that at the end of $\gamma_+$, $x_{2;4}$ is non-zero and purely imaginary. According to (13) if we start at $t = 0$, when returning to it after the loop $\gamma_+$ the only changes are due to the determination of $\log(1-t)$, which changes by $-2\pi\,\mathrm{i}$. Hence, the value of $x_{2;4}$ at the end of $\gamma_+$ is, except by normalizing factors, equal to $-\frac{1}{2}(a+2)\times(2\pi\,\mathrm{i})\times\xi_1(0)$ which is $\neq 0$ according to (12). In a similar way after the loop $\gamma_-$ the change is due to the determination of $\log(1+t)$ which cancels the one introduced by $\log(1-t)$. So, we get the identity for $\Phi_2$ after traveling along $\gamma_-\circ\gamma_+$. Furthermore, starting at $t = 0$, both $M_1^{\gamma_+}$ and $M_1^{\gamma_-}$ are unipotent. Hence, they are in $(G_1)^0$ (see [7], Proposition 2.2). In the case $p$ even, the proof follows similar steps. □

4. **Study of second and third order variational equations.** The variables in (9) satisfy the following equations

$$\begin{aligned} \dot{x}_{1;2,2} &= \tfrac{1}{1+\mu}x_{3;2,2}, & \dot{x}_{1;2,4} &= \tfrac{1}{1+\mu}x_{3;2,4}, & \dot{x}_{1;4,4} &= \tfrac{1}{1+\mu}x_{3;4,4}, \\ \dot{x}_{3;2,2} &= -\mathcal{D}_2, & \dot{x}_{3;2,4} &= -\mathcal{D}_M, & \dot{x}_{3;4,4} &= -\mathcal{D}_4, \end{aligned} \tag{17}$$

where

$$\mathcal{D}_2(t) := x_{2;2}^2(t) - 2r(t)\dot{x}_{2;2}^2(t), \quad \mathcal{D}_4(t) := x_{2;4}^2(t) - 2r(t)\dot{x}_{2;4}^2(t),$$

$$\mathcal{D}_M(t) := x_{2;2}(t)x_{2;4}(t) - 2r(t)\dot{x}_{2;2}(t)\dot{x}_{2;4}(t).$$

In a similar way, the variables in (10) involve the functions $\mathcal{D}_2(t), \mathcal{D}_4(t)$ and $\mathcal{D}_M(t)$. The equations are given in appendix 1.

The proof of Propositions 2 and 3 require the computation of some integrals along the path $\gamma_-\circ\gamma_+$. In what follows, given a function $f$ and a concrete determination $f(0)$ at $t = 0$, we shall denote by $f_+(z)$ the values it takes along $\gamma_+$ or along $\gamma_-$ when it changes in a continuous way from $t = 0$, and by $f_-(z)$ the values it takes along $\gamma_-$ after traveling along $\gamma_+$ and returning to $t = 0$, taking into account possible changes in the determination. Assume that $f_-(z) = f_+(z) + \hat{f}(z)$ for some $\hat{f}(z)$. Then $\int_{\gamma_-\circ\gamma_+} f(z)dz$ is equal to $-\int_{\gamma_+}\hat{f}(-z)dz$ if $f_+(z)$ is an even function (locally, around $t = 0$), and it is equal to $2\int_{\gamma_+} f_+(z)dz - \int_{\gamma_+}\hat{f}(-z)dz$ if $f_+(z)$ is odd.

**Proof of Proposition 2**. It is clear that $\mathcal{D}_2(t)$ involves only the functions $r(t)$, $\xi_1(t), \dot{\xi}_1(t)$. So, it is a polynomial. Moreover we recall that $x_{1;3}(t) = t/(1+\mu)$. Then

$$\int_\gamma \mathcal{D}_2(t)dt = 0, \qquad \int_\gamma x_{1;3}(t)\mathcal{D}_2(t)dt = 0$$

along any closed path $\gamma$. Therefore at the end of $\Gamma$ we have $x_{i;2,2} = 0$ for $i = 1, 3$. In a similar way $x_{4;1,2} = 0$ and $x_{4;2,3} = 0$ (see appendix 1).

Let us consider $\mathcal{D}_M(t)$. First we shall prove that

$$\int_{\gamma_+} \mathcal{D}_M(t)dt = 0.$$

Using the normalization $\xi_1(1) = 1$ and $\dot{\xi}(1) = a/4$ a simple computation shows that (see (13), (15) and (16))

$$\mathcal{D}_M(t) = -\frac{c_2}{2c_1}(a+2)(\log(1+t) - \log(1-t))\mathcal{D}_2(t) + h(t), \tag{18}$$

where $h(t)$ is a polynomial. We shall write $\mathcal{D}_2(t) = 2c_1^2 k(t)$ where

$$k(t) := \frac{1}{2}\xi_1^2(t) - \frac{1}{a}(1-t^2)\dot{\xi}_1^2(t). \tag{19}$$

**Lemma 3.** *The following identity holds:* $\int_0^1 k(t)dt = 0.$

*Proof.* Using integration by parts, (11) and (12) we have

$$\int_0^1 \left(\frac{1}{2}\xi_1^2(t) - \frac{1}{a}(1-t^2)\dot{\xi}_1^2(t)\right) dt =$$

$$\int_0^1 \frac{1}{2}\xi_1^2(t)dt - \frac{1}{a}(1-t^2)\dot{\xi}_1(t)\xi_1(t)|_0^1 + \int_0^1 \frac{1}{a}\xi_1(t)[(1-t^2)\ddot{\xi}_1(t) - 2t\xi_1(t)]dt =$$

$$\int_0^1 \left(\frac{1}{2}\xi_1^2(t) + \frac{1}{a}\left[-\frac{a}{2}\xi_1^2(t) + \frac{1-t^2}{2}\xi_1(t)\ddot{\xi}_1(t)\right]\right) dt = \frac{1}{2a}\int_0^1 (1-t^2)\xi_1(t)\ddot{\xi}_1(t)dt = 0.$$

Last equality follows because $\xi_1$ is an orthogonal polynomial in $[-1,1]$ with respect to the weight $1-t^2$, hence, orthogonal to $\ddot{\xi}_1$ and the product of both polynomials is an even function. □

Let $K(t)$ be the primitive of $k(t)$ such that $K(0) = 0$. Notice that after Lemma 3, $K(1) = 0$. Now we shall apply the following result

**Lemma 4.** *Let $g$ be a holomorphic function in a simply connected domain containing $\gamma_+$ and let $G$ be a primitive of $g$. Then* $\int_{\gamma_+} g(t)\log(1-t)dt = 2\pi\, i(G(1) - G(0)) = 2\pi\, i\int_0^1 g(t)dt$.

Using Lemmas 3 and 4 we have

$$\int_{\gamma_+} \log(1-t)\mathcal{D}_2(t)dt = 2c_1^2 \int_{\gamma_+} \log(1-t)k(t)dt = 0$$

and then, from (18), $\int_{\gamma_+} \mathcal{D}_M(t)dt = 0$.

The following expressions are easily obtained

$$\begin{aligned}
\mathcal{D}_{M,-}(t) &= \mathcal{D}_{M,+}(t) - \pi\,\mathrm{i}\frac{c_2}{c_1}(a+2)\mathcal{D}_2(t), \\
\mathcal{D}_{4,-}(t) &= \mathcal{D}_{4,+}(t) - 2\pi\,\mathrm{i}\frac{c_2}{c_1}(a+2)\mathcal{D}_{M,+}(t) - \pi^2\frac{c_2^2}{c_1^2}(a+2)^2\mathcal{D}_2(t), \quad (20)\\
x_{1;3}(t)\mathcal{D}_{M,-}(t) &= \frac{t}{1+\mu}\mathcal{D}_{M,+}(t) - \pi\,\mathrm{i}\frac{c_2}{c_1}\frac{(a+2)}{1+\mu}t\mathcal{D}_2(t),
\end{aligned}$$

where $\mathcal{D}_{M,+}(t)$ and $\mathcal{D}_{4,+}(t)$ are odd and even functions respectively. We recall also that $\mathcal{D}_2(t)$ is an even function. Therefore,

$$\int_{\gamma_-\circ\gamma_+} \mathcal{D}_M(t)dt, \quad \int_{\gamma_-\circ\gamma_+} \mathcal{D}_4(t)dt \quad \text{and} \quad \int_{\gamma_-\circ\gamma_+} x_{1;3}\mathcal{D}_M(t)dt$$

reduce to linear combinations of $\int_{\gamma_+} \mathcal{D}_M(t)dt$, $\int_{\gamma_+} \mathcal{D}_2(t)dt$ and $\int_{\gamma_+} t\mathcal{D}_2(t)dt$ which are equal to zero.

Furthermore $\gamma_-^{-1} \circ \gamma_+^{-1}$ is the complex conjugate of $\gamma_- \circ \gamma_+$. Moreover, we know that $\log(1+t) - \log(1-t)$ does not change determination after traveling through $\gamma_- \circ \gamma_+$. Then, the same is true for $\mathcal{D}_M, \mathcal{D}_4$ and $x_{1;3}\mathcal{D}_M$. Therefore $\int_\Gamma \mathcal{D}_M, \int_\Gamma \mathcal{D}_4$ and $\int_\Gamma x_{1;3}\mathcal{D}_M$ are all zero.

Using (17), (43) and (44) (see appendix 1) we get that after traveling along $\Gamma$ the following elements are zero

$$x_{1;2,4},\ x_{3;2,4},\ x_{3;4,4},\ x_{2;1,2},\ x_{2;2,3},\ x_{2;1,4},\ x_{4;1,4},\ x_{4;3,4}.$$

This ends the proof of Proposition 2. □

**Proof of Proposition 3**. We begin with the differential equations

$$\dot{x}_{2;2,2,4}(t) = r^{-2}(t)x_{4;2,2,4}(t) - 2r^{-3}(t)(2x_{4;2}(t)x_{1;2,4}(t) + x_{1;2,2}(t)x_{4;4}(t)),$$

$$\dot{x}_{4;2,2,4}(t) = -r(t)x_{2;2,2,4}(t) - 2x_{2;2}(t)x_{1;2,4}(t) - x_{2;4}(t)x_{1;2,2}(t) + r(t)x_{2;2}^2(t)x_{2;4}(t). \tag{21}$$

To solve previous equations and to obtain the values at the end of the path, we use the elementary "variation of the constants" method, which reduces the solution to compute quadratures, and requires also the final values of the first order variational equations at the end of the paths, which are obtained from Proposition 1. The same method will be used in appendix 1 to obtain the solutions of second order variational equations. After traveling along $\gamma_- \circ \gamma_+$ we get

$$x_{2;2,2,4} = \int_{\gamma_-\circ\gamma_+} (x_{1;2,2}(t)\mathcal{D}_4(t) + 2x_{1;2,4}(t)\mathcal{D}_M(t) - r(t)x_{2;2}^2(t)x_{2;4}^2(t))dt, \tag{22}$$

where $x_{1;2,2}(t), x_{1;2,4}(t)$, are the solutions of (17) with initial conditions $x_{1;2,2}(0) = 0$ and $x_{1;2,4}(0) = 0$. In a similar way we get

$$x_{4;2,4,4} = \int_{\gamma_-\circ\gamma_+} (-2x_{1;2,4}(t)\mathcal{D}_M(t) - x_{1;4,4}(t)\mathcal{D}_2(t) + r(t)x_{2;2}^2(t)x_{2;4}^2(t))dt. \tag{23}$$

To prove Proposition 3, it is sufficient to prove that the integrals in (22) and (23) are real and different from zero. From (22) and (23)

$$x_{2;2,2,4} + x_{4;2,4,4} = \int_{\gamma_-\circ\gamma_+} (x_{1;2,2}(t)\mathcal{D}_4(t) - x_{1;4,4}(t)\mathcal{D}_2(t))dt. \tag{24}$$

On the other hand, using (17), a simple computation shows that

$$x_{1;2,2}(t)\mathcal{D}_4(t) - x_{1;4,4}(t)\mathcal{D}_2(t) = \frac{d}{dt}[(-x_{1;2,2}(t)x_{3;4,4}(t) + x_{1;4,4}(t)x_{3;2,2}(t)].$$

We recall that, if $p$ is odd, $x_{1;2,2}(t)$ and $x_{3;2,2}(t)$ are polynomials equal to zero at $t = 0$. This implies that the primitive involved in (24) becomes null at both ends. Therefore $x_{2;2,2,4} + x_{4;2,4,4} = 0$ and it is sufficient to consider $x_{2;2,2,4}$.

We claim that the following relations hold (see appendix 2 for the proofs)

$$\int_{\gamma_-\circ\gamma_+} r(t)x_{2;2}^2(t)x_{2;4}^2(t)dt = 2\pi^2 c_1^2 c_2^2 (a+2)^2 \int_0^1 r(s)\xi_1^4(s)ds, \tag{25}$$

$$\int_{\gamma_-\circ\gamma_+} x_{1;2,2}(t)\mathcal{D}_4(t) = \int_{\gamma_-\circ\gamma_+} x_{1;2,4}(t)\mathcal{D}_M(t) = 8\pi^2 c_1^2 c_2^2 \frac{(a+2)^2}{1+\mu} \int_0^1 K^2(s)ds. \tag{26}$$

Therefore we obtain the following real expression for $x_{2;2,2,4}$

$$x_{2;2,2,4} = 24\pi^2 c_1^2 c_2^2 (a+2)^2 \int_0^1 \left[ \frac{1}{1+\mu} K^2(t) - \frac{1}{12a}(1-t^2)\xi_1^4(t) \right] dt.$$

Next lemma ends the proof of Proposition 3.

**Lemma 5.** *The following inequality holds for any $p > 2$*

$$Z := \int_0^1 \left[ \frac{1}{1+\mu} K^2(t) - \frac{1}{12a}(1-t^2)\xi_1^4(t) \right] dt > 0.$$

This lemma will be proved in the next section. We recall that $\xi_1(t)$ and $K(t)$ are polynomials of degrees $p-1$ and $2p-1$, respectively. So, for a given $p$, not too large, one can compute exactly the value of $Z$. To illustrate some of the difficulties that appear to prove that $Z > 0$, for arbitrary $p > 2$, we show first a couple of plots. Let us introduce

$$I_{\text{left}} = \int_0^1 \frac{1}{1+\mu} K^2(t)\, dt, \qquad I_{\text{right}} = \int_0^1 \frac{1}{12a}(1-t^2)\xi_1^4(t)\, dt, \tag{27}$$

both integrands being non-negative everywhere. Figure 4 left shows, for a moderate value $p = 9$, the function $\xi_1$, i.e., the Jacobi polynomial $P_8^{1,1}$ with the normalization introduced in Section 3 and also the functions $\frac{1}{1+\mu}K^2(t)$, $\frac{1}{12a}(1-t^2)\xi_1^4(t)$ after multiplication by a suitable constant to make them visible. One can observe that the dominant contributions to the integrals come from a narrow domain close to $t = 1$. We shall see in the proof that this domain is $\mathcal{O}(a^{-1})$ and it is essential for the proof. On the right part of the figure we display the ratio $R(p) = I_{\text{left}}/I_{\text{right}}$ as a function of $\log(a(p))$ up to $p = 3\,162$, the first value of $p$ for which $a(p) > 10^7$, recalling $a(p) = p^2 + p - 2$. The computations are done exactly (using PARI [2]) in rational arithmetic; some fractions require lots of digits. For instance, the integral in $Z$ multiplied by $1 + \mu$, requires up to $4\,273$ digits in the numerator and up to $4\,293$ in the denominator for $p$ up to $3\,162$. Values of $R(p)$ for small $p$ are shown in Table 1. One checks that for $p = 2$ the ratio is 1: both integrals are equal and cancel. One can also observe that $R(p)$ behaves almost linearly as a function of $\log(a(p))$. A fit suggests $R(p) \approx \alpha + \beta \log(a(p))$ with $\beta = 3/4$. We shall comment on this behaviour in Remark 6.

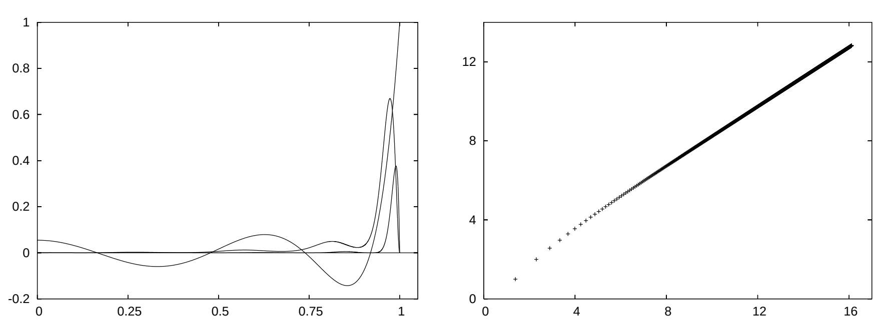


FIGURE 5. Left: The solution $\xi_1$ for $p = 9$ and the corresponding integrands in (27) multiplied by $5 \times 10^4$. The largest one is the term in $K^2$. Right: The ratio $R(p)$ vs $\log(a(p))$.

| $p$ | $R(p)$ | $p$ | $R(p)$ |
|---|---|---|---|
| 2 | 1 | 11 | 22759394912/5308411131 |
| 3 | 2 | 12 | 16777562286/3791260061 |
| 4 | 172/67 | 13 | 1301661512339/285994227707 |
| 5 | 663/223 | 14 | 3937129140486/843537994141 |
| 6 | 3952/1201 | 15 | 69209389132017/14494323713237 |
| 7 | 711763/200413 | 16 | 717917637801596207/147261612942021091 |
| 8 | 9537221/2527823 | 17 | 41008730304053787586/8253073211613057511 |
| 9 | 99832/25177 | 18 | 105957929821568901427/20952458480229302251 |
| 10 | 404790839/97889959 | 19 | 91823182181213244502/17863603738874767317 |

TABLE 1. The ratio $R(p) = I_{\text{left}}/I_{\text{right}}$ for small values of $p$.

5. **Proof of lemma 5.** Before proving (27) we sketch the steps to be followed.

- First we look at the limit behaviour of $\xi_1(t)$ near $t = 1$ as $p$ goes to infinity. A suitable scaling shows that they tend to the $J_1$ Bessel function (see, e.g., [1]).
- We introduce $t_2 = 1 - 4s_2/a \in [0, 1]$ where $s_2$ will be selected as a rational number close to the second positive zero of $J_1$. Then we write
$$\begin{aligned} Z \;\; > & \int_{t_2}^{1} \frac{1}{1+\mu} K^2(t)\, dt - \int_0^1 \frac{1}{12a}(1-t^2)\xi_1^4(t)\, dt = \\ & I_{\text{left}}^{[t_2,1]}(\xi_1) - (I_{\text{right}}^{[0,t_2]}(\xi_1) + I_{\text{right}}^{[t_2,1]}(\xi_1)), \end{aligned} \qquad (28)$$
where $I_{\text{left}}^{[t^*,t^{**}]}(\xi_1)$ denotes the integral $I_{\text{left}}$ but with the integration restricted to the interval $[t^*, t^{**}]$. The notation $I_{\text{right}}^{[t^*,t^{**}]}(\xi_1)$ has a similar meaning.
- The integrals on $[0, t_2]$ and $[t_2, 1]$ are bounded using different approximations of $\xi_1$.

**Lemma 6.** *Under the change of variables $t = 1 - 4s/a$ the functions $\xi_1(t)$, which depend on $p$, tend to a limit function $f(s)$ in any compact domain of the form $s \in [0, s_f]$ when $p \to \infty$. Furthermore the limit function satisfies $f(s) = J_1(\sqrt{8s})/\sqrt{2s}$, where $J_1$ denotes the Bessel function of first order.*

*Proof.* We recall that $\xi_1(t)$ is a polynomial solution of (11) of degree $p - 1$, which has been normalised so that $\xi_1(1) = 1$. The change of variables $t = 1 - 4s/a$ leads to the equation
$$\left(s - \frac{2}{a}s^2\right)\frac{d^2\xi_1}{ds^2} + \left(2 - \frac{8}{a}s\right)\frac{d\xi_1}{ds} + 2\xi_1 = 0, \qquad \xi_1(0) = 1, \qquad (29)$$
where we denote the new dependent variable as $\xi_1(s)$. Letting $a \to \infty$ we obtain, for bounded values of $s$,
$$s\frac{d^2 f}{ds^2} + 2\frac{df}{ds} + 2f = 0, \qquad f(0) = 1, \qquad (30)$$
where we denote as $f$ the limit function. Let $\xi_1(s) = \sum_{n\geq 0} b_n s^n$, $b_0 = 1$ and $f(s) = \sum_{n\geq 0} f_n s^n$, $f_0 = 1$ be the expansions of $\xi_1$ and $f$ around $s = 0$. From (29) and (30) we obtain the recurrences
$$b_{n+1} = -\frac{2(1 - n(n+3)/a)}{(n+1)(n+2)} b_n, \qquad f_{n+1} = -\frac{2}{(n+1)(n+2)} f_n, \qquad (31)$$

both of them to be compared later. In particular $f_n = \frac{(-2)^n}{n!(n+1)!}$. Introducing $\sigma = \sqrt{8s}$ it is immediate to identify $f(s) = J_1(\sigma)/\sqrt{2s}$. □

Note that $t = 0$ corresponds to $s = a/4$. Now we select a fixed value of $s$, for instance close to the second positive zero of $f$, $s_{\text{second}} \approx 929/151$. We take the value of $s$ as $s_2 = 929/151$ and then $t_2 = 1 - 4s_2/a$, depends on $a$. From now on we shall consider $p \geq 5$ in order to have $t_2 \in [0, 1]$.

Let us consider first the integrals in (28) on $[t_2, 1]$ and use the change $t = 1 - 4s/a$ introduced in Lemma 6. To simplify formulas, we shall keep the same notation $\xi_1(s)$ for the function $\xi_1$ expressed in terms of the variable $s$.

The idea is to approximate $\xi_1$ by the expansion to order 16 of $f$, that is $\tilde{f}(s) = \sum_{n=0}^{16} f_n s^n$. To bound the error we note that from (31) it follows that $|f(s) - \tilde{f}(s)|$ and $|df/ds(s) - d\tilde{f}/ds(s)|$ can be bounded in $[0, s_2]$ by $|f_{17}| \max\{s_2^{17}, 17s_2^{16}\} < 5 \times 10^{-12} =: \epsilon_1$. In a similar way, if we introduce $\tilde{\xi}_1(s) = \sum_{n=0}^{16} b_n s^n$, we get $|\xi_1(s) - \tilde{\xi}_1(s)| < \epsilon_1$ and $|d\xi_1/ds(s) - d\tilde{\xi}_1/ds(s)| < \epsilon_1$. Furthermore, it is clear that $\tilde{\xi}_1$ is a polynomial of degree 16 in $s$ and of degree 15 in $a^{-1}$ with rational coefficients. So, we can write $\tilde{\xi}_1(s) - \tilde{f}(s) = \sum_{k=1}^{15} a^{-k} Q_k(s)$ for some polynomials $Q_k(s)$ that can be computed using (exact) rational arithmetic. For $p \geq 3\,162$ we get

$$|\tilde{\xi}_1(s) - \tilde{f}(s)| < 1.9 \times 10^{-7}.$$

The same bound holds for $|\tilde{\xi}_1'(s) - \tilde{f}'(s)|$ where $'$ denotes $d/ds$. Then, for $s \in [0, s_2]$,

$$|\xi_1(s) - \tilde{f}(s)| < 2 \times 10^{-7} =: \epsilon, \qquad \left| \frac{d\xi_1}{ds}(s) - \frac{d\tilde{f}}{ds}(s) \right| < \epsilon. \tag{32}$$

Moreover, the following inequalities, to be used in the next lemmas, are trivial for $0 \leq s \leq s_2$

$$s\left|\tilde{f}\right| \leq 1, \quad s\left|\tilde{f}^2\right| \leq 1, \quad s\left|\tilde{f}^3\right| \leq 1, \tag{33}$$

$$\left|\tilde{f}\right| + s\left|\frac{d\tilde{f}}{ds}\right| \leq 1, \quad \tilde{f}^2 + s\frac{d\tilde{f}}{ds}^2 \leq 1. \tag{34}$$

Using the variable $s$ we can write

$$I_{\text{right}}^{[t_2,1]}(\xi_1) = \frac{8}{3a^3} \int_0^{s_2} s\left(1 - \frac{2s}{a}\right) \xi_1^4(s) ds < \frac{8}{3a^3} \int_0^{s_2} s\xi_1^4(s) ds =: J_{\text{right}}^{[0,s_2]}(\xi_1)$$

and

$$I_{\text{left}}^{[t_2,1]}(\xi_1) = \frac{8(a-2)}{a^4} \int_0^{s_2} \left[ \int_0^s (\xi_1(u)^2 - u(1 - 2u/a)\xi_1'(u)^2) du \right]^2 ds =: I_{\text{left}}^{[0,s_2]}(\xi_1).$$

We write some inequalities:

$$I_{\text{right}}^{[t_2,1]}(\xi_1) \leq J_{\text{right}}^{[0,s_2]}(\xi_1) \leq J_{\text{right}}^{[0,s_2]}(\tilde{f}) + |J_{\text{right}}^{[0,s_2]}(\xi_1) - J_{\text{right}}^{[0,s_2]}(\tilde{f})|, \tag{35}$$

$$I_{\text{left}}^{[t_2,1]}(\xi_1) \geq I_{\text{left}}^{[0,s_2]}(\tilde{f}) - |I_{\text{left}}^{[t_2,1]}(\xi_1) - I_{\text{left}}^{[0,s_2]}(\tilde{f})|, \tag{36}$$

where $I_*^*(\tilde{f})$ and $J_*^*(\tilde{f})$ are defined as the corresponding $I_*^*(\xi_1)$ and $J_*^*(\xi_1)$ replacing $\xi_1$ by $\tilde{f}$.

**Lemma 7.** *With the notation introduced before, the following bounds hold*

$$a^3 I_{left}^{[0,s_2]}(\tilde{f}) > M_l, \qquad a^3 J_{right}^{[0,s_2]}(\tilde{f}) < M_r,$$

*where $M_l$ and $M_r$ can be taken equal to 0.55555 and 0.13310 respectively.*

*Proof.* A symbolic manipulator (PARI) has been used to compute the integrals above using (exact) rational arithmetic. The values obtained, displaying only the first 10 decimal digits, are 0.5555528023... and 0.1330950485.... □

**Lemma 8.** *The differences of integrals using $\xi_1$ and $\tilde{f}$ are bounded as follows:*

$$a^3 |J_{right}^{[0,s_2]}(\xi_1) - J_{right}^{[0,s_2]}(\tilde{f})| \quad < \quad 0.2 \times 10^{-4} =: E_r,$$

$$a^3 |I_{left}^{[t_2,1]}(\xi_1) - I_{left}^{[0,s_2]}(\tilde{f})| \quad < \quad 0.5 \times 10^{-3} =: E_l.$$

*Proof.* We write $\xi_1(s) = \tilde{f}(s) + \delta(s)$. Using (32) we have $|\delta(s)| < \epsilon$ for $s \in [0, s_2]$. Then the inequalities (33) give, even using very rough estimates,

$$|J_{\text{right}}^{[0,s_2]}(\xi_1) - J_{\text{right}}^{[0,s_2]}(\tilde{f})| \leq \frac{8}{3a^3}\left((4\epsilon + 6\epsilon^2 + 4\epsilon^3)s_2 + \frac{1}{2}\epsilon^4 s_2^2\right) < 0.2 \times 10^{-4}.$$

For the left integral we write

$$|I_{\text{left}}^{[t_2,1]}(\xi_1) - I_{\text{left}}^{[0,s_2]}(\tilde{f})| = \frac{8}{a^3}(1 - 2/a)\left|\int_0^{s_2} K(s;\xi_1)^2 ds - \int_0^{s_2} K(s;\tilde{f})^2 ds\right|,$$

where

$$K(s;\xi_1) := \int_0^s \left[\xi_1(u)^2 - u(1 - 2u/a)\left(\frac{d\xi_1}{du}(u)\right)^2\right] du$$

and similar for $K(s;\tilde{f})$.

As before, using $\xi_1(s) = \tilde{f}(s) + \delta(s)$ and $\xi_1'(s) = \tilde{f}'(s) + \delta'(s)$, with $|\delta(s)|, |\delta'(s)| < \epsilon$ for $s \in [0, s_2]$ in $K(s;\xi_1)$, we get

$$K(s;\xi_1) = K(s;\tilde{f}) + A(s),$$

where

$$A(s) = \int_0^s \left[2\delta\tilde{f} + \delta^2 - u\left(1 - \frac{2u}{a}\right)(2\delta'\tilde{f}' + (\delta')^2)\right] du.$$

Therefore

$$|I_{\text{left}}^{[t_2,1]}(\xi_1) - I_{\text{left}}^{[0,s_2]}(\tilde{f})| = \frac{8}{a^3}(1 - 2/a)\left|\int_0^{s_2} (2A(s)K(s;\tilde{f}) + A(s)^2) ds\right|.$$

Moreover using (34) we obtain

$$\begin{aligned} |A(s)| \quad &\leq \quad \int_0^s \left(2\epsilon|\tilde{f}| + \epsilon^2 + u\left|1 - \frac{2u}{a}\right|(2\epsilon|\tilde{f}'| + \epsilon^2)\right) du \leq \\ &\int_0^s \left(\epsilon^2(1+u) + 2\epsilon(|\tilde{f}| + u|\tilde{f}'|)\right) du \leq 2\epsilon s + \epsilon^2\left(s + \frac{s^2}{2}\right). \end{aligned}$$

Then we obtain the bound

$$\int_0^{s_2} A(s)^2 ds \leq \frac{4}{3}\epsilon^2 s_2^3 + 4\epsilon^3\left(\frac{s_2^3}{3} + \frac{s_2^4}{8}\right) + \epsilon^4\left(\frac{s_2^3}{3} + \frac{s_2^4}{4} + \frac{s_2^5}{20}\right) < 2 \times 10^{-11} =: \Delta_1.$$

In a similar way we get $|K(s;\tilde{f})| \leq s$ for $s \in [0, s_2]$ and

$$\int_0^{s_2} |K(s;\tilde{f})A(s)| \leq \frac{2}{3}\epsilon s_2^3 + \epsilon^2\left(\frac{s_2^3}{3} + \frac{s_2^4}{8}\right) < 0.311 \times 10^{-4} =: \Delta_2.$$

Finally we obtain

$$a^3|I_{\rm left}^{[t_2,1]}(\xi_1) - I_{\rm left}^{[0,s_2]}(\tilde{f})| \quad \leq \quad 8(1-2/a)(2\Delta_2+\Delta_1) < 0.5\times 10^{-3}.$$

□

Let us consider now $I_{\rm right}^{[0,t_2]}(\xi_1)$. The following lemma provides an approximation for $\xi_1$ in $[0,t_2]$.

**Lemma 9.** *In the interval* $[0,t_2]$ *the function* $\xi_1(t)$ *is bounded by*

$$|\xi_1(t)| < \frac{1.69}{\sqrt{p-1}(p+1)}(1-t^2)^{-3/4}. \tag{37}$$

*Proof.* Let us introduce the new variables $z$ and $\theta$ in (11) as

$$z(t) = \xi_1(t)\left(1-t^2\right)^{3/4}, \quad \theta(t) = \left(n+\frac{3}{2}\right)\left(\frac{\pi}{2}-\arccos(t)\right), \tag{38}$$

where $n = p-1$ (see the beginning of Section 3). From (38) it follows

$$\frac{dz}{d\theta} = -y, \quad \frac{dy}{d\theta} = z - h(\theta)z, \quad h(\theta) = \frac{3}{(2n+3)^2\cos^2(\theta/(n+3/2))},$$

and introducing polar coordinates $z = R\cos(\gamma), y = R\sin(\gamma)$ and $\varphi = \gamma - \theta$ we reach the simple system

$$\frac{dR}{d\theta} = -\frac{1}{2}R\sin(2(\theta+\varphi))h(\theta), \quad \frac{d\varphi}{d\theta} = -\frac{1}{2}h(\theta)(1+\cos(2(\theta+\varphi))), \tag{39}$$

where $R_0 := R(t=0) = |\xi_1(0)|$, if we assume $p$ odd. For $p$ even some sin, cos functions are exchanged and then $R_0 = |\dot{\xi}_1(0)/(n+3/2)|$. In any case, $\varphi_0 := \varphi(t=0)$ is taken as 0 or $\pi$ in order to have $R_0\cos(\varphi_0) = \xi_1(0)$ if $p$ is odd, and $R_0\cos(\varphi_0) = -\dot{\xi}_1(0)/(n+3/2)$, if $p$ is even.

The equations (39) provide immediately

$$R_0\exp(-B(\theta)/2) \leq R(\theta) \leq R_0\exp(B(\theta)/2), \quad -B(\theta) \leq \varphi(\theta)-\varphi_0 \leq B(\theta), \tag{40}$$

where

$$B(\theta) = \int_0^\theta h(\tau)d\tau = \frac{3/2}{2n+3}\tan(\theta/(n+3/2)) \leq \frac{3/2}{2n+3}\frac{t_2}{\sqrt{1-t_2^2}} < \frac{6}{\sqrt{512 s_2}} =: \Delta,$$

and we have used $t_2 = 1-4s_2/a$.

Now we recover $\xi_1(t) = R(t)\cos(\varphi+\theta)(1-t^2)^{-3/4}$. Then

$$|\xi_1(t)| \leq R(t)(1-t^2)^{-3/4} \leq R_0\exp(\Delta/2)(1-t^2)^{-3/4}.$$

Using the bounds given in (12), for $|\xi_1(0)|$ and $|\dot{\xi}_1(0)|$, for odd and even $p$ respectively, we get

$$|\xi_1(t)| \leq \sqrt{8/\pi}\frac{1}{(p+1)\sqrt{p-1}}\exp(\Delta/2)(1-t^2)^{-3/4}$$

and (37) follows easily. □

**Remark 4.** Using also a rough bound from $d\varphi/d\theta$ as given in (40) we obtain variations of $\varphi$ bounded by 0.107, when expressing $\varphi$ as a function of $t$. It is also clear that for any fixed $t$, away from 1, one has bounds $\mathcal{O}(a^{-1/2})$. Better estimates on $R, \varphi$ and therefore on $\xi_1$, can be obtained using averaging to study the behaviour of the solutions of (39). See also remark 6.

**Remark 5.** The proof of Lemma 9 can be easily extended to arbitrary Gegenbauer polynomials $C_n^{(\alpha)}$ (see [1] for definition and properties), by introducing $\xi(t)(1-t^2)^{\alpha/2}, \theta=(n+\alpha)(\pi/2-\arccos(t))$.

**Lemma 10.** *The following bound holds for all* $p > 3\,162$

$$a^3\left|I_{right}^{[0,t_2]}(\xi_1)\right| < 0.01382 =: R_r.$$

*Proof.* From Lemma 9 one has to bound

$$\left|I_{\text{right}}^{[0,t_2]}(\xi_1)\right| < \int_0^{t_2}\frac{1}{12a}\left(\frac{1.69}{\sqrt{p-1}(p+1)}\right)^4\left(1-t^2\right)^{-2}dt.$$

Using that $\dfrac{1}{(p-1)^2(p+1)^4} < 1.00032a^{-3}$ holds for $p>3\,162$, the integral $\int_0^{t_2}(1-t^2)^{-2}dt = \dfrac{1}{4}\left(\dfrac{2t_2}{1-t_2^2}+\log\left(\dfrac{1+t_2}{1-t_2}\right)\right)$ and the explicit value $t_2=1-4s_2/a$, the lemma follows easily. □

Finally, from (28), (35), (36) and the above lemmas we can write

$$a^3Z \geq M_l - E_l - (R_r+M_r+E_r) > 0.4\,.$$

This ends the proof of Proposition 3 and therefore finishes the proof of Theorem 4. □

**Remark 6.** In fact, the neglected integral $I_{\text{left}}^{[0,t_2]}(\xi_1)$ has an important (positive) contribution to $Z$ in the sense that the ratio $I_{\text{left}}^{[0,1]}(\xi_1)/I_{\text{left}}^{[t_2,1]}(\xi_1)$ tends to $\infty$ when $p\to\infty$.

To give some idea about the claim above, let us replace $s_2$ by $s_m=A$ with a fixed value $A>>1$ and, hence, $A<<a$ for $a$ large enough. Then we replace $t_2$ by $t_m=1-4s_m/a$. We can derive an approximation for $\xi_1(t)$ in $[0,t_m]$ by using the same variables introduced in the proof of Lemma 9. Now $|\varphi(\theta)-\varphi(0)|<\Delta_m$ where $\Delta_m=6/\sqrt{512A}=\mathcal{O}(A^{-1/2})$ as follows from (40). In a similar way $R(\theta)=R_0(1+\mathcal{O}(A^{-1/2}))$ for $t\in[0,t_m]$.

Then we obtain the approximations

$$\xi_1(t)\approx R_0\cos(\varphi_0)\cos(\theta)(1-t^2)^{-3/4}\approx i_s\sqrt{\frac{8}{\pi}}a^{-3/4}\cos(\theta)(1-t^2)^{-3/4},$$

where we recall $\theta=(n+3/2)(\pi/2-\arccos(t))$ and $i_s$ is the sign of the dominant term in $\xi_1(s)$ (see (12)). The factor $\cos(\theta)$ has to be replaced by $\sin(\theta)$ for $p$ even. We assume $p$ odd in what follows.

Introducing $\psi=\theta/(n+3/2)$ we obtain the following approximation

$$\xi_1(\psi)\approx i_s\sqrt{\frac{8}{\pi}}a^{-3/4}(\cos(\psi))^{-3/2}\cos((n+3/2)\psi).$$

Using $\psi$ as independent variable the function $k$ introduced in (19) becomes, up to some constant

$$k(\psi)=\frac{1}{2}a^{-3/2}(\cos(\psi))^{-3}(\cos((n+3/2)\psi))^2- \tag{41}$$

$$a^{-5/2}\left[\frac{3}{2}(\cos(\psi))^{-5/2}\sin(\psi)\cos((n+3/2)\psi)-(n+\frac{3}{2})(\cos(\psi))^{-3/2}\sin((n+3/2)\psi)\right]^2.$$

As $n$ is large, when we integrate (41) one can replace $(\cos((n+3/2)\psi))^2$ and $(\sin((n+3/2)\psi))^2$ by the average value $1/2$ and $\sin((n+3/2)\psi)\cos((n+3/2)\psi)$ by zero. Furthermore, if $A$ is large enough, one can neglect the square of the first term inside $[*]^2$ in front of the square of the second one. Summarizing, we can approximate $k(\psi)$ by $-(1/4)a^{-3/2}(\cos(\psi))^{-3}$.

Let $\chi = \frac{\pi}{2} - \psi$. Then up to some constant, the left integral $I_{\text{left}}^{[0,t_m]}(\xi_1)$, can be written (up to some constant) as

$$\frac{1}{8a^3}\int_{\mathcal{O}(a^{-1/2})}^{\pi/2}\left(\int_{\chi}^{\pi/2}\frac{du}{(\sin(u))^2}\right)^2\sin(\chi)d\chi = \frac{1}{8a^3}\int_{\mathcal{O}(a^{-1/2})}^{\pi/2}\frac{\cos^2\chi}{\sin\chi}d\chi. \tag{42}$$

The dominant contribution to (42) comes from the domain $\chi = \mathcal{O}(a^{-1/2})$ and it is immediate that the result is $\mathcal{O}(\log(a))$. This proves the remark. Note that this also explains the results displayed in Figure 4 right. We do not state this result as a Proposition because, for shortness, the bounds of the errors in the application of averaging are not made explicit.

6. **Conclusions.** We have presented a very simple mechanical system for which, for some exceptional values of a mass ratio, a theoretical proof of non-integrability has been not possible with other available methods. A new approach, based on the use of higher order variational equations introduced in [10] or, equivalently, on the jet transport along a suitable chosen path in complex time, allows to establish the desired non-integrability result. The proof involves the use of different singularities and, hence, some amount of global information.

**Appendix 1.** For completeness in this appendix we give explicitly the second variational equations for the variables which appear in (10).

$$\begin{aligned}
\dot{x}_{2;1,2}(t) &= r^{-2}(t)x_{4;1,2}(t) - 2r^{-1}(t)\dot{x}_{2;2}(t),\\
\dot{x}_{4;1,2}(t) &= -r(t)x_{2;1,2}(t) - x_{2;2}(t),\\
\dot{x}_{2;1,4}(t) &= r^{-2}(t)x_{4;1,4}(t) - 2r^{-1}(t)\dot{x}_{2;4}(t),\\
\dot{x}_{4;1,4}(t) &= -r(t)x_{2;1,4}(t) - x_{2;4}(t),\\
\dot{x}_{2;2,3}(t) &= r^{-2}(t)x_{4;2,3}(t) - 2r^{-1}(t)x_{1;3}(t)\dot{x}_{2;2}(t),\\
\dot{x}_{4;2,3}(t) &= -r(t)x_{2;2,3}(t) - x_{1;3}(t)x_{2;2}(t),\\
\dot{x}_{2;3,4}(t) &= r^{-2}(t)x_{4;3,4}(t) - 2r^{-1}(t)x_{1;3}(t)\dot{x}_{2;4}(t),\\
\dot{x}_{4;3,4}(t) &= -r(t)x_{2;3,4}(t) - x_{1;3}(t)x_{2;4}(t).
\end{aligned}$$

See equations (7), (11), (17) and (21) for the other variational equations of first, second and third order used in the proof.

Using standard methods, as explained in the proof of Proposition 3, we obtain the following values after traveling along $\gamma_- \circ \gamma_+$

$$\begin{aligned}
x_{2;1,2} &= \int_{\gamma_-\circ\gamma_+}\mathcal{D}_M(t)dt, & x_{2;1,4} &= \int_{\gamma_-\circ\gamma_+}\mathcal{D}_4(t)dt,\\
x_{4;1,2} &= \int_{\gamma_-\circ\gamma_+}-\mathcal{D}_2(t)dt, & x_{4;1,4} &= \int_{\gamma_-\circ\gamma_+}-\mathcal{D}_M(t)dt,
\end{aligned} \tag{43}$$

$$
\begin{aligned}
x_{2;2,3} &= \int_{\gamma_-\circ\gamma_+} x_{1;3}(t)\mathcal{D}_M(t)dt, \qquad & x_{2;3,4} &= \int_{\gamma_-\circ\gamma_+} x_{1;3}(t)\mathcal{D}_4(t)dt, \\
x_{4;2,3} &= \int_{\gamma_-\circ\gamma_+} -x_{1;3}(t)\mathcal{D}_2(t)dt, \qquad & x_{4;3,4} &= \int_{\gamma_-\circ\gamma_+} -x_{1;3}(t)\mathcal{D}_M(t)dt.
\end{aligned} \tag{44}
$$

**Appendix 2.** In this appendix we give the details for the proof of (25) and (26). Using the notation introduced in Section 4 we get, $\xi_{2,-}(t) = \xi_{2,+}(t) - \pi\,\mathrm{i}(a+2)\xi_1(t)$, and

$$
r(t)\xi_1^2(t)\xi_{2,-}^2(t) = r(t)\xi_1^2(t)\xi_{2,+}^2(t) + \pi(a+2)r(t)\xi_1^3(t)(-2\,\mathrm{i}\xi_{2,+}(t) - \pi(a+2)\xi_1(t)),
$$

where $r(t)\xi_1^2(t)\xi_{2,+}^2(t)$ is an even function of $t$. Then

$$
\begin{aligned}
\int_{\gamma_-\circ\gamma_+} r(t)\xi_1^2(t)\xi_2^2(t)dt &= -\pi(a+2)\int_{\gamma_+} r(-t)\xi_1^3(-t)[-2\,\mathrm{i}\xi_{2,+}(-t) - \pi(a+2)\xi_1(-t)]dt \\
&= -2\pi\,\mathrm{i}(a+2)\int_{\gamma_+} r(t)\xi_1^3(t)\xi_{2,+}(t)dt = -\pi\,\mathrm{i}(a+2)^2\int_{\gamma_+}\log(1-t)r(t)\xi_1^4(t)dt.
\end{aligned}
$$

Now using Lemma 4 we obtain the following identity, proving claim (25) in Section 4

$$
\int_{\gamma_-\circ\gamma_+} r(t)\xi_1^2(t)\xi_2^2(t)dt = 2\pi^2(a+2)^2\int_0^1 r(t)\xi_1^4(t)dt.
$$

Let us consider now the first integral in (26). From (20) we obtain

$$
\begin{aligned}
x_{1;2,2}(t)\mathcal{D}_{4,-}(t) \quad &= \quad x_{1;2,2}(t)\mathcal{D}_{4,+}(t) - 2\pi\,\mathrm{i}\frac{c_2}{c_1}(a+2)x_{1;2,2}(t)\mathcal{D}_{M,+}(t) - \\
&\qquad \frac{c_2^2}{c_1^2}\pi^2(a+2)^2 x_{1;2,2}(t)\mathcal{D}_2(t),
\end{aligned}
$$

where $x_{1;2,2}(t)\mathcal{D}_{4,+}(t)$ is an even function. Then

$$
\begin{aligned}
\int_{\gamma_-\circ\gamma_+} x_{1;2,2}(t)\mathcal{D}_4(t) \quad &= \quad 2\pi\,\mathrm{i}\frac{c_2}{c_1}(a+2)\int_{\gamma_+} x_{1;2,2}(-t)\mathcal{D}_{M,+}(-t)dt + \\
&\qquad \frac{c_2^2}{c_1^2}\pi^2(a+2)^2\int_{\gamma_+} x_{1;2,2}(-t)\mathcal{D}_2(-t)dt.
\end{aligned}
$$

The second integral above is equal to zero because $x_{1;2,2}(t)$ and $\mathcal{D}_2(t)$ are polynomials. Using (18) and Lemma 4

$$
\begin{aligned}
\int_{\gamma_+} x_{1;2,2}(t)\mathcal{D}_{M,+}(t)dt \quad &= \quad \frac{c_2}{2c_1}(a+2)\int_{\gamma_+} x_{1;2,2}(t)\mathcal{D}_2(t)\log(1-t)dt = \\
&\qquad \frac{c_2}{c_1}(a+2)\pi\,\mathrm{i}\int_0^1 x_{1;2,2}(t)\mathcal{D}_2(t)dt.
\end{aligned}
$$

Taking into account that $\mathcal{D}_2(t) = 2c_1^2 k(t)$ and

$$
x_{1;2,2}(t) = -\frac{2c_1^2}{1+\mu}\int_0^t K(s)ds
$$

we obtain

$$
\int_{\gamma_+} x_{1;2,2}(t)\mathcal{D}_{M,+}(t)dt = 4\pi\,\mathrm{i}c_1^3 c_2\frac{(a+2)}{1+\mu}\int_0^1 K^2(t)dt. \tag{45}
$$

Therefore we obtain finally

$$\int_{\gamma_-\circ\gamma_+} x_{1;2,2}(t)\mathcal{D}_4(t) = 8\pi^2 c_1^2 c_2^2 \frac{(a+2)^2}{1+\mu}\int_0^1 K^2(t)dt.$$

This proves the required expression for the first integral in (26).

In a similar way it is not difficult to see that

$$\int_{\gamma_-\circ\gamma_+} x_{1;2,4}(t)\mathcal{D}_M(t) = -\pi\,\mathrm{i}\frac{c_2}{c_1}(a+2)\left[\int_{\gamma_+} x_{1;2,2}(t)\mathcal{D}_{M,+}(t) + \int_{\gamma_+} x_{1;2,4}(t)\mathcal{D}_2(t)dt\right].$$

Now we use an argument similar to the one used to prove that the expression in (24) is zero. From

$$x_{1;2,2}(t)\mathcal{D}_M(t) - x_{1;2,4}(t)\mathcal{D}_2(t) = \frac{d}{dt}[-x_{1;2,2}(t)x_{3;2,4}(t) + x_{1;2,4}(t)x_{3;2,2}(t)]$$

it follows

$$\int_{\gamma_+}(x_{1;2,2}(t)\mathcal{D}_M(t) - x_{1;2,4}(t)\mathcal{D}_2(t))dt = 0$$

and, hence,

$$\int_{\gamma_-\circ\gamma_+} x_{1;2,4}(t)\mathcal{D}_M(t) = -2\pi\,\mathrm{i}\frac{c_2}{c_1}(a+2)\int_{\gamma_+} x_{1;2,2}(t)\mathcal{D}_{M,+}(t)dt$$

which proves the second part of (26) using (45).

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