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Minimal topological chaos coexisting with a finite set of homoclinic and periodic orbits

Walter Huaraca [a], Valentín Mendoza [b,*]

[a] DMA-UFV, Universidade Federal de Viçosa, Campus Universitário, CEP 36570-900, Viçosa, MG, Brazil

[b] Instituto de Matemática e Computação, Universidade Federal de Itajubá, Av. BPS 1303, Bairro Pinheirinho, CEP 37500-903, Itajubá, MG, Brazil

HIGHLIGHTS

- The pruning method can be applied to certain physical models.
- The combinatorics of the pruning map is found uncrossing invariant manifolds.
- Infinite pruning regions are related to singularities without rotation.

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ABSTRACT

In this note we explain how to find the minimal topological chaos relative to finite set of homoclinic and periodic orbits. The main tool is the pruning method, which is used for finding a hyperbolic map, obtained uncrossing pieces of the invariant manifolds, whose basic set contains all orbits forced by the finite set under consideration. Then we will show applications related to transport phenomena and to the problem of determining the orbits structure coexisting with a finite number of periodic orbits arising from the bouncing ball model.

1. Introduction

By minimal topological chaos relative to a homoclinic orbit P we mean the minimal structure of orbits that a system containing this homoclinic orbit can have in its isotopy class. It was Poincaré who realizes that the existence of such orbits implies a higher complexity [1], and Birkhoff and Smale proved that, under regular conditions, there are infinitely many periodic orbits in every neighbourhood of P [2–5].

It is known that a non-autonomous perturbation of an integrable system, satisfying Melnikov's conditions, creates homoclinic orbits with transversal intersection and also at least a chaotic set having a dense set of periodic orbits. See Fig. 1. Such models have many applications going from transport phenomena [6], the analysis of bifurcations in a driver oscillator [7] to the dynamics of bubbles in time-periodic straining flows [8]. In all these applications a natural question is the following: *which is the minimal periodic orbits structure that a map, having P as a homoclinic orbit, can have*? The same question can be formulated if P is a finite set of homoclinic and periodic orbits since chaotic behaviour can be created from the finite set of topological shapes induced by P. In [9] and references there in, periodic orbits are studied in applications to laser models, Lorentz and Rössler attractors, the Belousov–Zhabotinskii reaction, etc. To answer that question we need the notion of *forcing* introduced by P. Boyland.

Let f be a homeomorphism on the disk and let P be an orbit of f. The isotopy class of (P, f) is given by its braid type which identifies all the orbits that are equivalent to P under isotopies [10]. We say that (P, f) *forces* an orbit Q if every homeomorphism g isotopic to f relative to P, having an orbit with the braid type of P, must also has an orbit with the braid type of Q. The set of all the orbits whose braid types are forced by an orbit (P, f) will be denoted by Σ_P. Thus Σ_P contains a topological representative of each orbit that is forced by P, and it shows us the minimal description of the set of periodic orbits that a map can have given only a topological data.

One of the first result about the forcing relation of homoclinic orbits was stated by Handel in [11]. He provides conditions for

* Corresponding author.
E-mail addresses: walterhv@ufv.br (W. Huaraca), valentin@unifei.edu.br (V. Mendoza).

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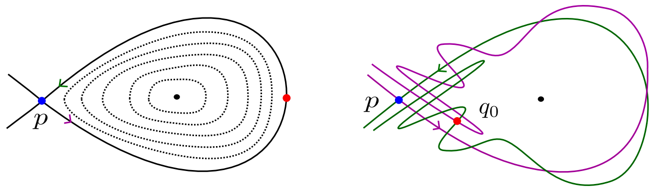

Fig. 1. Homoclinic orbit appearing after a non-autonomous perturbation of an integrable system.

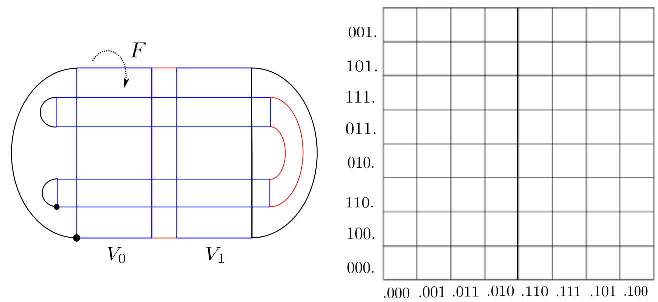

Fig. 2. The Smale horseshoe and its symbol square.

ensuring that a finite set of homoclinic orbits imply the existence of a fixed point. In Hulme's thesis [12] there exists an extension of the Bestvina–Handel algorithm [13] which can be used for computing an efficient graph map or a generalized pseudoanosov representative within the isotopy class of a homoclinic orbit.

In [14–16] Collins has proposed a method for determining a graph representative whose orbits represent the dynamics forced by the homoclinic orbit P and, under certain conditions, construct a diffeomorphism that minimizes the topological entropy the isotopy class relative to P. This is done studying a trellis, a part of the homoclinic tangle of P. A similar motivation was given in [17] by Mitchell and Delos, where the attention was towards into the escape segments by iterations of the map.

All these methods can find exact or approximated symbolic dynamics in Σ_P but unfortunately the number of symbols is always increased as the trellis becomes more and more complicated and a computational cost is needed. Another disadvantage is that, except in a few cases, it is not clear how to apply them to the study of an infinitely many family of homoclinic orbits.

In [18] a pruning method is proposed for finding, given a homoclinic orbit, an Axiom A diffeomorphism whose non-wandering set realizes all the braid types forced by that orbit. This method can be considered as a differentiable version of the pruning theory developed by de Carvalho [19] for pruning surfaces homeomorphisms, and can be extended for finding Σ_P rel to a finite set of homoclinic and periodic orbits, since Σ_P is actually the *complement* of the pruning region rel to P. In [20] the technique was used for organizing certain horseshoe periodic orbits by forcing.

In fact, in this note we will explain how the pruning method works if P consists of certain infinite families of homoclinic orbits found in transport phenomena by Rom-Kedar in [21,22]. It will be showed, up isotopies, the pruning region rel to these orbits. Furthermore the method will be applied to a finite set of periodic orbits which include those ones studied by Tufillaro in [23] for the bouncing ball model, who has proposed a pruning region joining invariant manifolds. We improve his pruning region showing the existence of a map that realizes it which was not proved in [23]. We should note that the lines followed in this work can be adapted to a wide range of sets of periodic and homoclinic orbits arising from experimental data.

2. A model for minimal chaos

Our working model is the Smale horseshoe [5] which was one the first examples exhibiting deterministic chaos. This is a diffeomorphism F acting on a sub-disk of the disk as in Fig. 2. F is an Axiom A map, that is, F has hyperbolic structure on its non-wandering set which consists of an attractor point within the left semi-disk and a Cantor set K contained in the union of the rectangles $V_0 \cup V_1$. Then it was proved that F restricted to K is conjugated to the shift σ on the two-symbols compact space $\Sigma_2 = \{0, 1\}^{\mathbb{Z}}$. More general properties of Axiom A maps can be found in [24]. Collapsing segments joining two boundary points it is obtained the symbol square [25] represented in Fig. 2 as well.

We only devote our study to horseshoe homoclinic orbits of the form $q_0 = {}^{\infty}0.1w10^{\infty}$, where w is a finite word of symbols 0's and 1's, that is, homoclinic orbits at the intersection of the stable and unstable manifolds of the fixed point with code 0^{∞}. These orbits often appear in dynamical applications in a wide range of systems as this one in Fig. 1.

Now we recall the pruning ideas proposed by Cvitanović in [26]. He has observed that certain dynamical systems are better understood if we consider them as incomplete or pruned horseshoes. This means that certain systems can be obtained from the uncrossing of pieces of the invariant manifolds of the Smale horseshoe or an another well-known Axiom A map. The regions where orbits were eliminated are called *pruning regions*. So the symbolic dynamics of the system corresponds to the symbolic dynamics of the horseshoe except the orbits included inside the pruning region. This powerful idea simplifies the orbit analysis since it is sufficient to find a good pruning region in order to describe the orbits structure.

Several authors as [25,27–30] have followed the pruning approach, and their results were directed to find rules for the remaining symbol dynamics, but no illumination was provided about how invariant manifolds influence the final grammar.

A pruning formalism was given in [19] by de Carvalho for pruning, in particular, the horseshoe F. It demands the existence of a pruning domain, that is, a topological simply connected domain D bounded by two segments θ_s and θ_u which belong to the stable manifold and the unstable manifold of periodic points, respectively. Then D is called a pruning domain if it satisfies the following condition:

$$F^n(\theta_s) \cap \mathrm{Int}(D) = \emptyset = F^{-n}(\theta_u) \cap \mathrm{Int}(D), \quad \forall n \geq 1. \tag{1}$$

Thus the pruning theorem [19] claims that condition (1) is sufficient for eliminating all orbit within $\mathrm{Int}(D)$ in the sense that an isotopy of F can be implemented in such a way that there are no recurrent points in $\mathrm{Int}(D)$ for the homeomorphism G at the end of the isotopy. As a consequence the non-trivial dynamics of G are given by σ on $\Sigma_2 \setminus \cup_{i\in\mathbb{Z}} F^i(\mathrm{Int}(D))$. Because this theorem reigns in the topological level in which there is not notion of invariant manifolds, this is not applicable to Cvitanović's pruning approach.

To solve that impasse one of us has proposed, in a joint work with A. de Carvalho [31], a differentiable version of the pruning theorem, that is used to prune Axiom A maps since hyperbolic structure allows us to make G, the end of the pruning isotopy, an Axiom A map too, although the most important property to point out is that this pruning isotopy uncrosses invariant manifolds in a controlled manner which means that uncrossings only happen in the interior of D and its iterates. See [18] for the details.

Recalling that a *bigon* $\mathcal{I}$ is a simply connected domain bounded by a segment of a stable manifold and a segment of an unstable manifold, it was proved in [18] that, given a homoclinic orbit P, Σ_P can be found eliminating all the bigons of F relative to P by successive prunings. Fig. 3 shows the elimination of a bigon $\mathcal{I}$ under the effect to the uncrossing of the invariant manifolds within D by a pruning isotopy.

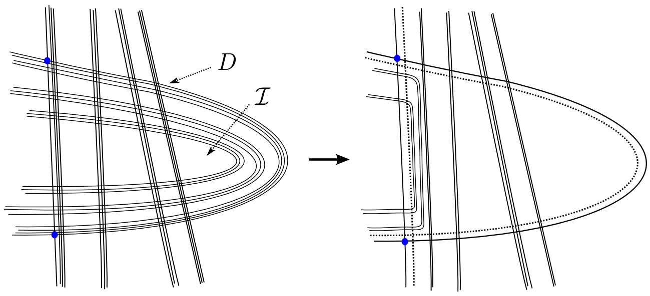

Fig. 3. Eliminating a bigon $\mathcal{I}$ within a pruning domain D.

More precisely it was proved that if the number of pruning domains, relative to P and necessary to eliminate all the bigons, is finite, then the dynamics of the final pruning map ψ_P associated to all the pruning domains, called the *hyperbolic pruning map relative to P*, characterizes Σ_P. It is done using a generalization of a persistence theorem given by Handel in [32]. Thus if $\{D_1, \ldots, D_n\}$ is the set of pruning domains with that property then

$$\Sigma_P = \Sigma_2 \setminus \cup_{i\in\mathbb{Z}} \sigma^i(\cup_{k=1}^{n} \mathrm{Int}(D_k)) \qquad (2)$$

up a finite number of boundary periodic points. So Σ_P is a subshift of finite type joint to a finite number of attractors. The equality (2) can be understood saying that the orbits forced by P are these ones which do not intersect the *pruning region* $\mathcal{P} = \cup_{k=1}^{n} \mathrm{Int}(D_k)$. Although the results in [18] are defined for only one homoclinic orbit, they can be easily adapted to a finite set $P = \{P_1, \ldots, P_l\}$ of periodic or homoclinic orbits, providing in the latter case that Σ_P is transitive. If the $P_1, \ldots, P_l$ are homoclinic ones to the same fixed point then it is possible to prove that the hyperbolic pruning map is always transitive on Σ_P.

So every homeomorphism f on the disk, containing a infinite orbit with the braid type of P, must has an invariant set Λ such that $f|_\Lambda$ is semiconjugated to $\sigma|_{\Sigma_P}$, so Σ_P describes the minimal orbit structure relative to P that such f can exhibit. Actually the semiconjugacy preserves the braid types. Thus the set $\mathrm{BT}(\Lambda, f)$ is equal to $\mathrm{BT}(\Sigma_P, F)$. Hence a lower bound for topological entropy of f can be calculated by the asymptotics of $\frac{1}{n}\ln(|\mathrm{Per}_n \cap \Sigma_P|)$ where Per_n denotes the set of periodic orbits of period n in Σ_2.

Since Σ_P is subshift of finite type, there exists a Markov partition for the state space despite the fact that the homoclinic orbit has the braid type of a homoclinic tangency. Generating partitions with homoclinic tangencies as boundary were constructed by Grassberger and Kantz in [33] although it was not possible to define the set of primary homoclinic tangencies. A criterion based in the analysis of the curvature was given for doing that in [34]. Instead using those properties, our method only needs the topology of the embedding of the orbits on an Axiom A map. Thus a finite pruning region implies that there exists a Markov partition for the dynamics up a finite number of boundary periodic points. The main open problem of our approach is related to the possibility that the number of pruning domains, needed for eliminating all the bigons, be infinite, that is, whenever the elimination of a bigon implies the creation of another and so, ad infinitum. Examples will be given in [20] and in Section 4 where a technique for leading that limit case will be sketched and a possible explanation for that phenomena will be presented.

Actually using bigons for determining forcing relations is not new in dynamical systems. In [35] T. Hall has associated maps without bigons to horseshoe periodic orbits. His *non-bogus transition* property can be understood in the pruning point of view as the non-existence of bigons. By an application of the Bestvina–Handel's algorithm, he was enable of finding Σ_P if $P = P_q$ is a quasi-one-dimensional orbit, that is, if the code of P_q is c_{q1}^0, for some $q \in (0, 1/2] \cap \mathbb{Q}$, where c_q is a palindromic word of 0's and 1's symbols obtained by the following rule: If $q = m/n$ is lowest terms, the word c_q is $10^{k_1}1^20^{k_2}1^2 \cdots 1^20^{k_m}1$ where $k_1 = \lfloor 1/q \rfloor - 1$ and $k_i = \lfloor i/q \rfloor - \lfloor (i-1)/q \rfloor - 2$ for $2 \leqslant i \leqslant m$ ($\lfloor x \rfloor$ is the greatest integer which does not exceed x). See also [36]. In this case, only one pruning domain is needed for determining Σ_{P_q}: the domain D_q bounded by a stable segment $\theta_s \subset W^s((c_q0)^\infty)$ and an unstable segment $\theta_u \subset W^u(0^\infty)$ which intersect at the heteroclinic points ${}^\infty 0.(c_q0)^\infty$ and ${}^\infty 01.(c_q0)^\infty$. It implies that the periodic orbits forced by P_q are all orbits which are smaller than P_q in the unimodal order $\geqslant_1$. As one can inferred from Sections below, the Hall's word c_q has became crucial for the forcing on horseshoe braids.

3. Applications to physical phenomena

Now we will show the pruning domains needed for finding Σ_P for certain homoclinic orbits. They were introduced by Easton in [37] and were associated to transport phenomena by Rom-Kedar in [21,22].

The first ones are the orbits called type-$\{l, 0, 0, 0\}$ which have the form $E_l = {}^\infty 0.10^l 10^\infty$ for certain positive integer l. It is not difficult to prove that they satisfy the hypothesis given in [37,21]. These orbits are a particular case of *star* homoclinic orbits which have the form $P_0^q = {}^\infty 0.c_q 0^\infty$ where c_q is the Hall's word defined above. Thus one can see that E_l corresponds P_0^q with $q = \frac{1}{l+1}$.

In [18] it was also proved that star homoclinic orbits demand only one pruning domain D_q for eliminating the bigons. That domain is bounded by a segment of the stable manifold of $\sigma^2(P_0^q)$ and a segment of the unstable manifold of the fixed point 1^∞ which intersect at the points ${}^\infty 1.0^{l-1}10^\infty$ and ${}^\infty 10.0^{l-1}10^\infty$. See Fig. 4 for an example with $l = 3$.

The pruning map G associated to D_q has invariant manifolds which look like these ones in Fig. 4. A symbolic representation of Σ_{E_l} appears in Fig. 5.

The second ones correspond to the homoclinic tangle called type-$\{l, m, k, 0\}$ by Rom-Kedar, who have showed numerical evidence in [22] to claim that they arise naturally in transport phenomena. We will suppose that $m \geq l$ and $k \geq l$. The Rom-Kedar's conditions [22] imply that the type $(l, m, k, 0)$ homoclinic tangle is the same than this one defined by the homoclinic orbits $A_{l,m} = {}^\infty 0.10^{l-1}110^{m-1}10^\infty$ and $B_{k,l} = {}^\infty 0.10^{k-1}110^{l-1}10^\infty$. There are two subcases to be considered.

Case I. If $m = l$ then $A_{l,l} = {}^\infty 0.10^{l-1}110^{l-1}10^\infty$ is again a star homoclinic orbit. We can see that $A_{l,l} = P_0^q$ with $q = \frac{2}{2l+3}$. Thus only one pruning domain is needed for destroying the bigons and since $B_{k,l}$ is always included in $\Sigma_{A_{l,l}}$ it follows that $A_{l,l}$ forces the existence of $B_{k,l}$, for any $k \geq l$.

Case II. If $m > l$ then two pruning domains are needed: a domain D_1 defined by a stable segment passing through $\sigma^2(A_{l,m})$ and an unstable segment passing through 1^∞ which intersect at the points ${}^\infty 10.0^{k-2}110^{l-1}10^\infty$ and ${}^\infty 1.0^{k-2}110^{l-1}10^\infty$; and a domain D_2 defined by a vertical segment joining the points ${}^\infty 010^{k-1}110^{l-1}.10^{m-1}10^\infty$ and ${}^\infty 010^{k-1}110^{l-2}1.10^{m-1}10^\infty$ which belongs to the stable manifold of ${}^\infty 010^{l-1}1.10^{m-1}10^\infty$, and an unstable segment passing through ${}^\infty 010^{k-1}110^{l-1}.10^\infty$. The reader is encouraged to prove that these two domains are sufficient to our purposes. Thus its pruning region is $\mathcal{P}_{l,m,k} = \mathrm{Int}(D_1) \cup \mathrm{Int}(D_2)$. Fig. 6 shows the domains for the values $l = 3$, $m = 4$ and $k = 5$, and the orbits forced by the homoclinic tangle type-$\{3, 4, 5, 0\}$.

Furthermore one can observe the following:

- If $m, k \to \infty$ then $A_{l,m} \to 0^\infty.10^{l-1}110^\infty$ and $B_{k,l} \to 0^\infty 110^{l-1}10^\infty$ which are clearly equivalent to E_l fact that was noted in [21].

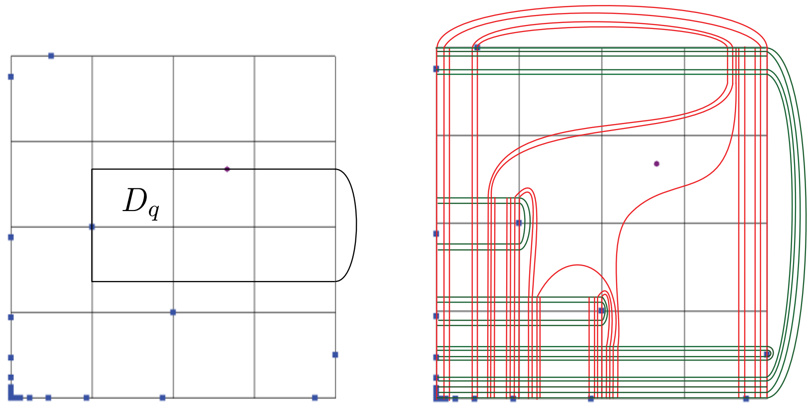

Fig. 4. The pruning domain associated to $E_3 = {}^{\infty}0.100010^{\infty}$ and its pruning diffeomorphism.

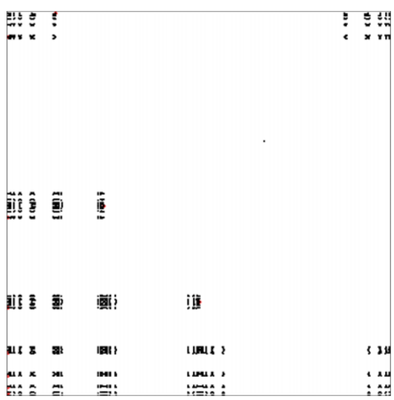

Fig. 5. A symbolic representation of Σ_P with $P = E_3 = {}^{\infty}0.100010^{\infty}$.

- The case $k = m$ is particularly important in applications to area-preserving maps [22]. If $l < l'$ then, for $m \geq l$ and $m' \geq l'$, we have $\mathrm{Orb}(A_{l',m'}) \cap \mathcal{P}_{l,m,m} = \emptyset$ and $\mathrm{Orb}(B_{m',l'}) \cap \mathcal{P}_{l,m,m} = \emptyset$; thus, by (2), the homoclinic tangle type-$\{l, m, m, 0\}$ forces the existence of all the orbits of the homoclinic tangle type-$\{l', m', m', 0\}$. By the same reasons one can conclude that, if $m < m'$, the homoclinic tangle type-$\{l, m, m, 0\}$ forces the existence of all orbits of the homoclinic tangle type-$\{l, m', m', 0\}$. It proves that the topological entropy is monotonically decreasing with l and m, which is consistent with the numericals showed in Tables 1 and 2 of [22].

4. Pruning relative to periodic orbits

Maybe the most important property of the pruning method is that it unifies the analysis of the forcing relation of homoclinic and periodic orbits. Thus as we have said before if $P = \{P_1, P_2, \ldots, P_l\}$ is a finite set of periodic or homoclinic orbits then the set Σ_P of orbits whose braid types are forced by P is obtained eliminating all the bigons relative to P. This can useful for experimental approaches when it is only possible to extract a finite set of orbits from a physical model [23,38–42]. Usually the main step to compute forcing implications of orbits extracted from experimental data is to find the *basis set* which is a finite set of orbits that are not forcing related and that forces all the periodic orbits obtained from the numericals [43]. Thus the basis set gives us all the chaotic information that the system can have providing a pruning region relative to it. Here we will calculate pruning regions associated to certain basis sets.

Now we will study orbits arising from the bouncing ball system, a model that has been extensively studied in the literature in physics, see for instance [44,45] and references there in. As Tufillaro has numerically observed in [23], the horseshoe orbits P_1, P_2 and P_3 with codes 10110111, 101101011 and 101111010, respectively, define the basis set of a bouncing ball model up period 11. He has proposed a pruning region joining these points by stable and unstable leaves, but such construction does not have a dynamical meaning in the sense that it is not possible to realize if that pruning region corresponds to a homeomorphism of the disk. Here we will define a pruning region formed by domains satisfying condition (1). Thus we are going to construct a sequence of pruning domains $D_1, D_2, \ldots$ aiming the elimination of the bigons by pruning isotopies. In every step k, the orbits already eliminated will be the orbits contained in $\cup_{j=1}^{k} \mathrm{Int}(D_j)$. Since this process is infinite, the final pruning map will be no longer an Axiom A map, but its non-wandering set will have all the orbits forced by the given ones $\{P_i\}_{i=1}^{3}$.

By the Hall's notation, $P_1 = R_{3/8}$ is a rotation of angle $\frac{3}{8}(2\pi)$ around the fixed point 1^{∞}, P_2 has code $c_{2/5}011$ and P_3 has code

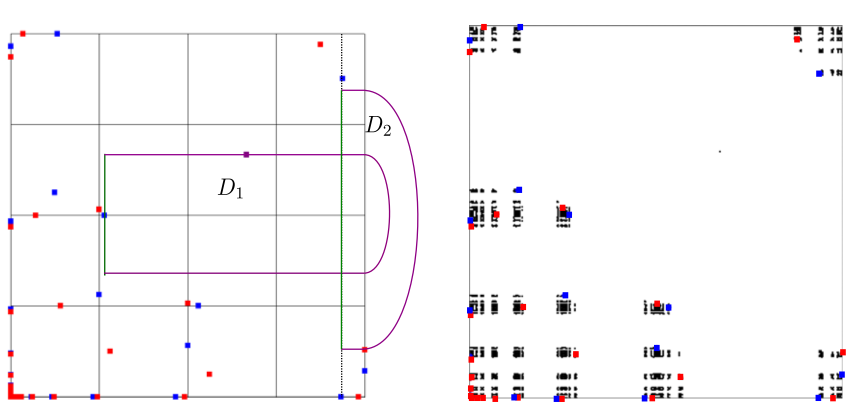

Fig. 6. The pruning region relative to $A_{3,4}$ and $B_{5,3}$ and the set of orbits forced by the homoclinic tangle type-$\{3, 4, 5, 0\}$.

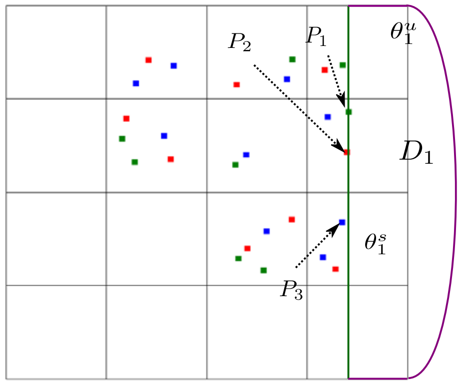

Fig. 7. The pruning domain D_1. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

$c_{3/7}0$. Fig. 7 shows these orbits in green, red and blue colours, respectively, in the symbol plane. These orbits, except the rotation, also were considered in [46, Table IV] for analysing chaotic signals of driven laser.

In fact, $\{P_1, P_2, P_3\} = P_{3/8,2/5,3/7}$ is a particular case of the triplet $P_{r,q,q'} = \{R_r, Q_q, P_{q'}\}$ where R_r is a rotation of angle $2r\pi$, Q_q has code c_q011 and $P_{q'}$ has code $c_{q'}0$ with $r, q, q' \in \mathbb{Q} \cap (0, 1/2)$ and $q < q'$. Thus we are going to construct the pruning region rel $P_{r,q,q'}$. So our construction works for a countably many family of triplets.

These type of triplets also appear in [47] by Letellier et al. where $P_{5/11,3/7,4/9}$ was found in a Rössler attractor (See period 11 orbits of [47, Table I]); in that case, since $P_{4/9}$ forces $R_{5/11}$, it is sufficient to study only $Q_{3/7}$ and $P_{4/9}$. Since c_q011 and c_q110 are codes of orbits with the same braid type [48], the triplet $P_{5/11,3/7,4/9}$ also appears in the spectrum of orbits obtained from the dynamics of a vibrating string in [41, Table I]. Noting that the pruning region that will be constructed does not lead to an one-dimensional dynamics, maybe our method could explain why certain periodic orbits are missing in the periodic spectra of the experimental data found in [41].

We only consider the case $r < q$. Let M and N be the periods of Q_q and $P_{q'}$, respectively. By the definition of c_q it follows that $c_{q'}0 \leqslant_1 c_q011 \leqslant_1 R_r$ when projected to the lower unstable leaf of the horseshoe. Then the first pruning domain D_1 is defined by a stable leaf θ_1^s passing through R_r and going from ${}^{\infty}0.(R_r)^{\infty}$ to ${}^{\infty}1.(R_r)^{\infty}$, and an unstable segment θ_1^u joining the same points. See Fig. 7.

Pruning D_1 from the Smale horseshoe, one can obtain an Axiom A map ψ_1. By Theorem 17 of [49], R_r is both a stable and an unstable boundary point for ψ_1. So the configuration of the invariant manifolds structure of ψ_1 looks like this one in Fig. 8 which is a blow up of the section $[0.7, 0.95] \times [0, 1]$ of the symbol plane. The map ψ_1 has a bigon which can be extended to a pruning domain D_2 bounded by a stable segment θ_2^s containing Q_q and a segment θ_2^u included in the unstable manifold of some point of the orbit of R_r. If one prune ψ_1 using the domain D_2, it is obtained an Axiom A map ψ_2 whose invariant manifolds look like Fig. 9. Note that the orbits eliminated by ψ_2, that is, the orbits which fall in $\mathrm{Int}(D_1) \cup \mathrm{Int}(D_2)$, are these ones whose codes are bigger than the code of Q_q, except R_r. Then the basic set of ψ_2 is

$$\{S : S \leqslant_1 Q_q\} \cup \{R_r\}.$$

Since $c_q \geqslant_1 c_{q'}$, one can construct a pruning domain D_3 bounded by a segment θ_3^s containing $(11c_q0)^{\infty}$ and an unstable segment θ_3^u containing $P_{q'}$, as in Fig. 9. So θ_3^s and θ_3^u intersect at the points ${}^{\infty}(c_{q'}0).(11c_q0)^{\infty}$ and ${}^{\infty}(c_{q'}0)c_{q'}1.(11c_q0)^{\infty}$. Hence $D_3 = \{x.y : (11c_q0)^{\infty} \leqslant_1 x \leqslant_1 (c_q011)^{\infty}, (0c_{q'})^{\infty} \leqslant_1 y \leqslant_1 1c_{q'}(0c_{q'})^{\infty}\}$.

Uncrossing the invariant manifolds inside D_3 by a pruning, we obtain an Axiom A map ψ_3 whose invariant manifolds are as in Fig. 10. Note that ψ_3 still has a bigon $\mathcal{I}$ contained within a pruning domain D_4. By the analysis of the $\psi_3^{-N}(\theta_3^u)$ it follows

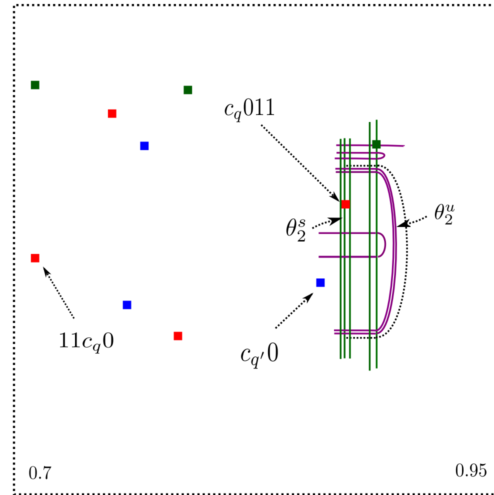

Fig. 8. Invariant manifolds structure of ψ_1.

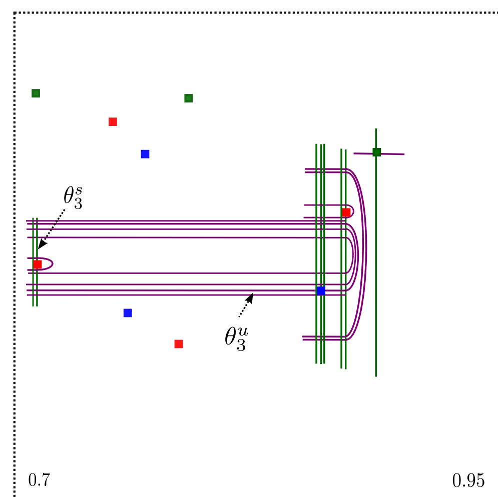

Fig. 9. Invariant manifolds structure of ψ_2.

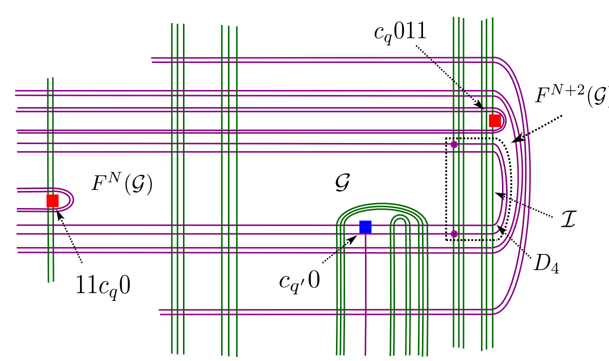

Fig. 10. The regions $\mathcal{G}$ and $\mathcal{R}_4$(in dotted lines).

that D_4 is contained in the region $\mathcal{R}_4$ defined by the points $A_4 = {}^{\infty}(c_{q'}0)11.c_{q'}1(11c_q0)^{\infty} B_4 = {}^{\infty}(c_{q'}0)10.c_{q'}1(11c_q0)^{\infty}$, that is, $\mathcal{R}_4 = \{x.y : c_{q'}1(11c_q0)^{\infty} \leqslant_1 x \leqslant_1 (c_q011)^{\infty}, 01(0c_{q'})^{\infty} \leqslant_1 y \leqslant_1 11(0c_{q'})^{\infty}\}$. The stable boundary of D_4 belongs also to the boundary of a region $\mathcal{G}$ which is limited by three stable leaves and three unstable leaves. By the combinatorics of A_4 and B_4 we see that the $(N + 2)$th iterate of $\mathcal{G}$ has as frontier a segment of the unstable boundary of D_4. See Fig. 10.

Applying one more time the pruning method, one can uncross the invariant manifolds that are inside D_4. Making the construction

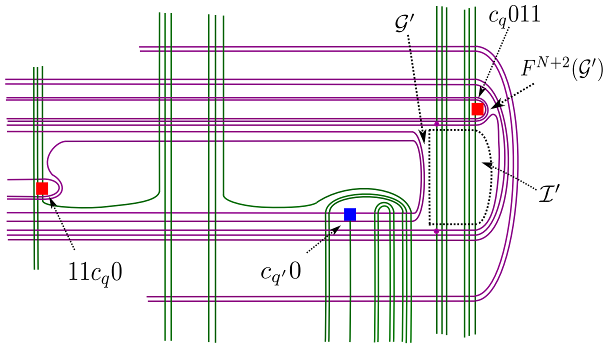

Fig. 11. Pruning map ψ_4 associated to D_4.

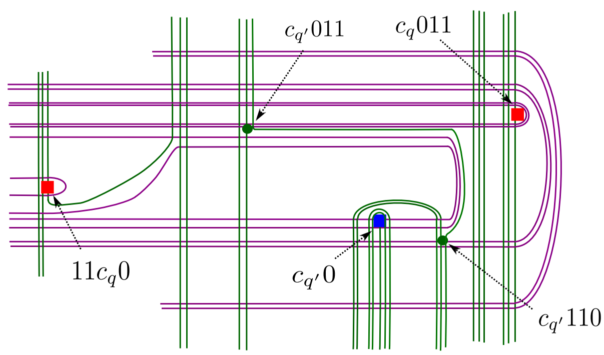

Fig. 12. The limit map ψ_∞.

of ψ_4, the pruning diffeomorphism associated to D_4, the bigon $\mathcal{I}$ and the region $\mathcal{G}$ are eliminated and they are substituted by a new bigon $\mathcal{I}'$ and a new region $\mathcal{G}'$ which maintain the same properties than $\mathcal{I}$ and $\mathcal{G}$. See Fig. 11. So there exists a pruning domain D_5 containing $\mathcal{I}'$. The domain D_5 is *asymmetric* in relation to the central horizontal line.

A combinatorial argument proves that D_5 is included in a region $\mathcal{R}_5$ defined by the points $A_5 = {}^\infty(c_{q'}0)11c_{q'}011.c_{q'}1(11c_q0)^\infty B_5 = {}^\infty(c_{q'}0)10c_{q'}110.c_{q'}1(11c_q0)^\infty$. The domain D_5 has the same properties than D_4 and hence one can repeat the process for finding a new pruning domain D_6 included within a region $\mathcal{R}_6$ defined by the points $A_6 = {}^\infty(c_{q'}0)11(c_{q'}011)^2.c_{q'}1(11c_q0)^\infty$ and $B_6 = {}^\infty(c_{q'}0)10(c_{q'}110)^2.c_{q'}1(11c_q0)^\infty$. Note that $D_4 \subset D_5 \subset D_6$.

Proceeding inductively in that way, we can find a increasing sequence of *asymmetric* pruning domains D_j, with $D_i \subset D_j$ if $5 \leq i < j$, and regions $\mathcal{R}_j$ defined by the points

$$A_j = {}^\infty(c_{q'}0)11(c_{q'}011)^{j-4}.c_{q'}1(11c_q0)^\infty$$

and

$$B_j = {}^\infty(c_{q'}0)10(c_{q'}110)^{j-4}.c_{q'}1(11c_q0)^\infty.$$

Hence we will obtain a sequence of pruning maps ψ_j associated to D_j. After pruning all these domains we obtain a homeomorphism ψ_∞ which is no longer an Axiom A map, but whose combinatorics can be described by the pruning region

$$\mathcal{P}_{r,q,q'} = \cup_{i=1}^{3} \operatorname{Int}(D_i) \cup \operatorname{Int}(D_\infty),$$

where D_∞ is included inside the region $\mathcal{R}_\infty = \{x.y : c_{q'}1(11c_q0)^\infty \leqslant_1 x \leqslant_1 (c_q011)^\infty, (011c_{q'})^\infty \leqslant_1 y \leqslant_1 (110c_{q'})^\infty\}$. See Fig. 12.

Thus ψ_∞ has an invariant set $K = \Sigma_2 \setminus \cup_{i\in\mathbb{Z}} \sigma^i(\mathcal{P}_{r,q,q'})$ given by

$$K = \{R_r\} \cup \{S : S \leqslant_1 Q_q \quad \text{and} \quad S \cap (\operatorname{Int}(D_3) \cup \mathcal{R}_\infty) = \emptyset\}$$

which is non-uniformly hyperbolic in all its points except in two of them with period $(N+2)$: $(c_{q'}110)^\infty$ and $(c_{q'}011)^\infty$. Finally, one has that $\Sigma_{P_{r,q,q'}} = K$ up a finite number of boundary periodic points. This set has been represented in the symbol plane in Fig. 13 up to orbits of period 19.

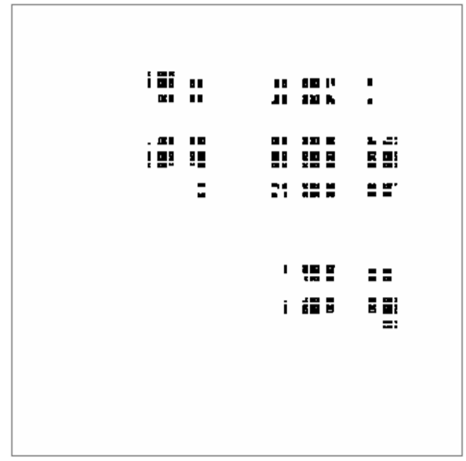

Fig. 13. The set K in the symbol plane.

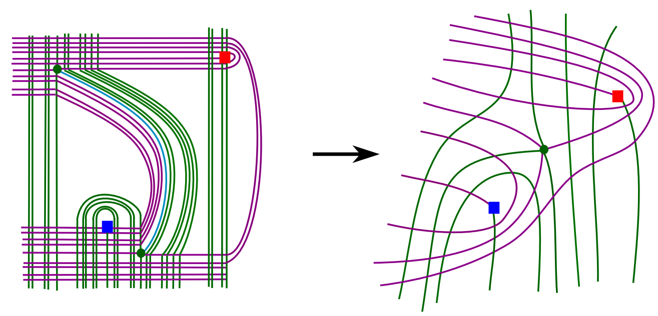

Fig. 14. A 3-pronged singularity.

Now we will argue a possible explanation for the necessity of infinite pruning domains for pruning relative to certain basis sets. In our example, *collapsing* the wandering pieces of $W^s(K) \cup W^u(K)$, ψ_∞ projects to a pseudo-Anosov map ϕ with a finite number of singularities, and the orbits of $(c_{q'}110)^\infty$ and $(c_{q'}011)^\infty$ become a unique orbit of period $N+2$ that is a 3-pronged singularity without rotation. A schematic representation of these points is pictured in Fig. 14.

This collapsing process, that was introduced by Bonatti and Jeandenans on Axiom A maps [24, Chapter 8], is devoted to find the minimal Nielsen–Thurston's representative ϕ within the isotopy class of ψ_∞, the main ingredient for determining the minimal structure of periodic orbits. As de Carvalho and Hall have observed, whenever ϕ has a n-pronged singularity with rotation 0, one needs asymmetric pruning domains [50, Section 4.6.1]. It seems that symmetric pruning domain only create pronged singularities with non-null rotation, and that, given a set of periodic orbits, only a finite number of symmetric pruning domains can be constructed. So what we have seen in this paper and in many other examples for which we have implemented the pruning method (see the final Section of [20]) is that if a n-pronged singularity of ϕ has rotation 0 then it is needed an *infinite* number of *asymmetric pruning domains*.

5. Conclusion

Identifying a finite set of homoclinic or periodic orbits P with horseshoe orbits we can try to find the set of pruning domains that are necessary to eliminate the bigons of the horseshoe rel those orbits. If that set is finite then the orbits forced have a representative in Σ_P. Thus Σ_P is the minimal topological chaotic set coexisting with P.

It seems to be true that if the minimal representative rel to a set of orbits has a *n*-pronged singularity without rotation then we need a infinite number of asymmetric pruning domains, but nowadays there is no a proof for that observation. But, even if the set of pruning domains is infinite, there exist cases, as the examples in Section 4, where Σ_P is characterized by these pruning domains building a limit map that is a non Axiom A model of the minimal dynamics. Maybe a reason for that is the fact that the *combinatorics* of the asymmetric domains is the same, so at least a symbolic description of the missing orbits can be calculated. So it will be interesting to prove if one of the following implications (or their reverses) is true: rotation 0 $\Longrightarrow$ asymmetric pruning domains $\Longrightarrow$ infinite pruning domains.

There is not restriction on the type of Axiom A maps that one can prune. As for the horseshoe template, the pruning method can be useful to the topological organization of periodic orbits coming from Axiom A maps with more than two symbols as these ones contained in [47,51]. In these cases a good information about the full symbolic dynamics of the template and of the positions of the bigons is necessary.

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