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TOPOLOGICAL ORGANIZATION OF (LOW-DIMENSIONAL) CHAOS

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ABSTRACT. Recent progress toward classifying low-dimensional chaos measured from time series data is described. This classification theory assigns a template to the time series once the time series is embedded in three dimensions. The template describes the primary folding and stretching mechanisms of phase space responsible for the chaotic motion. Topological invariants of the unstable periodic orbits in the closure of the strange set are calculated from the (reconstructed) template. These topological invariants must be consistent with any model put forth to describe the time series data, and are useful in invalidating (or gaining confidence in) any model intended to describe the dynamical system generating the time series.

Statistical measures and topological methods are the two major types of analysis used when studying chaos in smooth dynamical systems. These two approaches, the statistical and topological, often give us different information about the same dynamical system [Fr]. The ergodic (statistical) theory of dissipative dynamical systems focuses its attention on an invariant measure $\mu(\Omega)$ defined on the invariant limit set Ω (i.e., a strange attractor or repeller) [Ec]. Information about an invariant measure can have many useful applications. In time series analysis, for instance, $\mu(\Omega)$ is an essential ingredient in building nonlinear predictive models directly from time series [Ge].

Topological methods of smooth dynamical systems theory are also of great value in time series analysis. In particular, in the context of low-dimensional chaos, topological techniques allow us to develop a classification theory for chaotic invariant limit sets. In addition, topological properties often put strong constraints on the dynamics (for instance, the existence or non-existence of certain orbits [Ha]). A topological analysis is also an essential ingredient for developing rapidly convergent calculations of the metric properties of the attractor [Cv].

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Therefore, when analyzing a time series from a chaotic dynamical system we advocate a two step procedure. First, analyze the topological organization of the invariant set, and second dress this topological form with its metric structure. We believe, at least in context of low-dimensional chaos, that as much information as possible should be gleaned from the topology the chaotic limit set as a first step toward modeling the dynamics. This topological information plays at least two important roles in applications to time series analysis. First, topological invariants can be used to identify (or invalidate) models put forth to explain the data, and second, the topological classification of chaotic sets serves as a promising first step in developing predictive models of nonlinear time series data.

Recently, this topological approach to time series analysis has been worked out in great detail in the context of chaotic invariant sets of "low-dimensional" flows. In this article, by "low-dimensional" we mean flows in R^n with invariant sets of dimension less than or equal to 3, i.e., systems with one unstable direction (one positive Lyapunov exponent). By restricting our attention to this class of systems, it is possible to develop a rather complete physical theory for the topological classification of such systems and to develop practical algorithms for applying this classification scheme to time series data from experiments. In this article we will review work on this classification theory. For recent efforts on applying this classification theory to modeling the dynamics we refer the reader to a review article by Mindlin and Gilmore [Mi1] which also contains many practical details about topological time series analysis. For an elementary introduction to the knot theory and dynamical systems background appropriate for this article see Reference [Tu1].

The major device in this analysis is the template (or knot-holder) of the hyperbolic chaotic limit set [Ho]. Roughly, a template is an expanding map on a branched surface. A low-dimensional chaotic limit set with one unstable direction has a rich set of recurrence properties which are determined by the unstable saddle periodic orbits embedded within the strange set. These unstable periodic orbits provide a sort of skeleton on which the strange attractor rests. For flows in three dimensions, these periodic orbits are closed curves, or knots. The knotting and linking of these periodic orbits is a bifurcation invariant, and hence these simple topological invariants can be used to identify or "fingerprint" a strange attractor [Mi2, Tu2]. Templates are central to this analysis because periodic orbits from a three-dimensional flow of a hyperbolic dynamical system can be placed on a template in such a way as to preserve their original topological structure. Thus templates provide a visualizable model for the topological organization of the chaotic limit sets. Templates can also be describe algebraically by finite matrices and this in turn gives us a quantitative classification theory describing the primary folding and stretching structure of the strange set [Tu1].

The strategy behind the template theory is as follows. For a nonlinear dynamical system there are generally two regimes that are well understood, the regime where a finite number of periodic orbits exists and the hyperbolic regime of fully developed chaos. The essential idea is to reconstruct the form of the fully developed chaotic limit set from a non-fully developed (possibly non-hyperbolic) region in parameter space. Once the hyperbolic limit set is identified, then the topological information gleaned from the hyperbolic limit set can be used to make predictions about the chaotic limit set in other (possibly non-hyperbolic) parameter regimes, since topological invariants such as knot types, linking numbers, and relative rotation rates [So1, So2] are robust under parameter changes.

The identification of a template from a chaotic time series of low dimension proceeds in five steps [Mi3, Mi1]: search for close returns, three-dimensional embedding of the time series, calculation of topological invariants, template identification, and template verification.

In the first step, the search for close returns [Au, Tu2], the time series is examined for subsegments of the data which almost return to themselves after n-cycles. These subsegments of the time series are taken as surrogates for the unstable (saddle) period-n orbits which exist in the closure of the strange set. This search for close returns (unstable periodic orbits) can be done either before or after the time series is embedded in a three-dimensional space [Mi3].

The next step is to embed the time series in a three-dimensional space. Developing an embedding procedure which "optimizes" the topological information in the time series is the key to success with the topological analysis of time series data. In principal there are several candidates for an embedding procedure. Both the method of delays [Pa], and an embedding based on a singular value decomposition analysis are reasonable choices and are described by D. Broomhead in these proceedings. As a practical matter great care must be taken to see that the embedding procedure eliminates any (parametric) drift in the data (for instance, this may by accomplished by judicious filtering), and that the embedding procedure also seeks to maximize the geometric spatial separation of the embedded time series trajectory. With these two criteria in mind, Mindlin and Gilmore [Mi3] have developed a "differential phase space embedding" which works remarkably well for their analysis of data from the Belousov-Zhabotinskii reaction. On a case by case basis, finding an embedding which "optimizes" the extraction of topological information inherent within the (experimental) time series does not pose a major obstacle to the analysis. Rather it suggests that a lot of good work is yet to be done in developing a new branch of engineering which might be dubbed "topological signal processing."

In the embedded space, topological invariants (linking numbers, relative rotation rates, and braid words) of the surrogate periodic orbits found in the first step can be calculated. Just a few of these suffice to determine a template [Mi2, Mi3, Me]. In fact, one can also identify the template by examining the stretching and folding of points on the strange attractor as it evolves through one full cycle [Mc, Le], and also by examining the "line-diagram" of a few geometric braids calculated from the embedded periodic orbits[Ha]. Thus, the form of the template is usually very much over determined by the available experimental data. The fact the the template is determined from a (small) finite amount of information should come as no surprise. Each template is nothing but a geometric picture for the suspension of a full shift hyperbolic symbol system which we formally associate to the (possibly non-hyperbolic) chaotic time series. This full shift system has the same basic folding and stretching structure of the original flow, and it might even be found in the original (experimental) system in a parameter regime where a chaotic repeller exists.

Once identified, the template can be used to calculate an additional (infinite) set of topological invariants including (self) rotation rates, (self) linking numbers, knot types, polynomial invariants, and so on. If the template identification is correct, these invariants must all agree with those found in the time series data. If these invariants do not agree we can reject the proposed template. If they all agree, we get added confidence that the template is correctly identified. These topological invariants must also agree with any set of differential equations or other dynamical model proposed for the data. Thus, this gives us a way of falsifying (or gaining confidence in) any proposed model.

Each template itself is equivalent to a "framed braid" [Me]. A framed braid is just a geometric braid with an integer associate to each strand called the framing. The linking of this framed braid is described by a framed braid linking matrix, and it is this (finite) matrix which we take as our quantitative (integer) characterization for the topology of the strange set. For more details with an abundance of pictures see Chapter 5 of Reference [Tu1].

The template characterization and classification has recently been applied to a wide variety of time series data from experimental systems including the Belousov-Zhabotinskii chemical clock [Mi3], a laser with a saturable absorber [Pap], an NMR-laser [Tu3], and a CO_2 laser with modulated losses [Le].

The template classification theory is just the beginning of topological time series analysis. There are many directions now to take this work. Perhaps the most promising is exploiting the connection between certain braid types (periodic orbits) and complex behavior in the flow supporting this braid type. Since Thurston's work in the 70's on braid types and dynamics on the punctured disk, it has been know that the existence of certain types of braids (i.e., the so called pseudo-Anosov ones) are sufficient to imply that a dynamical system has positive topological entropy, that is, that the system is chaotic [Th]. Mindlin and Gilmore found such a braid type (periodic orbit) in their analysis of the Belousov-Zhabotinskii reaction [Mi3]. It is the period-7 pretzel knot of the horseshoe with symbolic name 0110101. The existence of this single "non-well ordered orbit" orbit [Ga] allows Mindlin and Gilmore to conclude that the system is chaotic (at least in the topological sense meaning the existence of an infinite number of periodic orbits forming a complex chain recurrent set) without calculating any Lyapunov exponents or fractal dimensions.

Indeed, as emphasized by D. Broomhead in these proceedings, some of the most exciting work in nonlinear dynamics is the current close interplay between mathematics and experimental physics. In essence one can seek, in doing an experiment, to show that certain mathematical hypothesis hold in the given experimental configuration. If these mathematical hypothesis can be experimentally verified, then one can learn much more about the system then either statistical inference or physical experimentation alone would provide.

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