Braid analysis of (low-dimensional) chaos

Nicholas B. Tufillaro*
Center for Nonlinear Studies and T13, MS-B258
Los Alamos National Laboratory, Los Alamos, NM 87545 USA
(27 June 1993)

We show how to calculate orbit implications (based on Nielsen-Thurston theory) for horseshoe type maps arising in chaotic time series data. This analysis is applied to data from the Belousov-Zhabotinskii reaction and allows us to (i) predict the existence of orbits of arbitrarily high periods from a finite amount of time series data, (ii) calculate a lower bound to the topological entropy, and (iii) establish a "topological model" of a system directly from an experimental time series.

05.45.+b, 47.52.+j

I. INTRODUCTION

Braids arise as periodic orbits in dynamical systems modeled by three-dimensional flows (see Fig. 1) [1–4]. The existence of a single periodic orbit of a dynamical system can imply the coexistence of many other periodic orbits. The most well-known example of this phenomenon occurs in the field of one-dimensional dynamics and is described by Sarkovskii's Theorem [5]. Less well-known is the fact that analogous results hold for two-dimensional systems [3]. In one-dimensional dynamics it is useful to study the period (or the permutation) of an orbit [6]. In two-dimensional systems it is useful to study the *braid type* of an orbit [2]. Given this specification, we can ask whether or not the existence of a given braid (periodic orbit) *forces* the existence of another; as in the one-dimensional case, algorithms have recently been developed for answering this question [7–9].

As originally observed by Auerbach et. al., unstable periodic orbits are available in abundance from a single chaotic time series using the method of close recurrence [10,11]. By a "braid analysis" we propose to analyze a chaotic time series by first extracting an (incomplete) spectrum of periodic orbits, and second ordering the extracted orbits according to their orbit forcing relationship. As shown in this paper, it is often possible to find a single periodic orbit (or a small collection of orbits) which forces many orbits in the observed spectrum. This single orbit also forces additional orbits of arbitrarily high period. This analysis is restricted to "low-dimensional" flows (roughly, flows which can be modeled by systems with one unstable Lyapunov exponent), however it has a strong predictive capability.

We would also like to point out that this analysis gives us an effective and mathematically well defined "pruning procedure" for chaotic two-dimensional diffeomorphisms [12]. Instead of asking for rules describing which orbits are missing (pruned), we instead look for those orbits which must be present. For low period orbits (say up to period 10) this procedure can predict all those orbits which must be present in the flow. This procedure will usually miss orbits of higher period, however from an experimental viewpoint the low period orbits are the most important and accessible. Orbits of low period often force an infinity of other orbits. This is illustrated in one-dimensional dynamics by the famous statement "period three implies chaos" [13]. An analogous statement in two-dimensional dynamics is that a non-well-ordered period three braid implies chaos [14].

This paper is organized as follows. In section II we briefly review the theory of orbit forcing in one- and two-dimensional maps (three-dimensional flows). In section III we show how the braid analysis method works by applying it to times series data generated from the Rossler equations. This section also discusses some useful numerical and symbolic refinements to the method of close recurrence. In section IV we apply the braid analysis method to data from the Belousov-Zhabotinskii reaction. Our analysis builds directly on the original topological analysis of this data set due to Mindlin et. al. [15]. In section V we offer some concluding remarks by indicating how this method can be extended to low-dissipation and conservative systems. In the examples studied in this paper we do have good control of the symbolics. In principle, though, this method does not require good control of the symbolics (a good partition) and can thus overcome some of the current difficulties associated with finding good symbolic descriptions for (nonhyperbolic) strange attractors [16].

II. ORBIT FORCING THEORY

In this paper we study in detail the orbit structure of horseshoe type attractors. In particular, we will make a good deal of use of those features of one-dimensional dynamics which must always carry over to two dimensions [17]. Therefore, let us begin by summarizing some well-established [6,18] definitions and results pertaining to the periodic orbit structure of maps of the line. Let $F : \mathbf{R} \to \mathbf{R}$ be a continuous map, and $R = \{x_1, \ldots, x_n\}$ be a period n orbit of F with its points labeled in such a way that $x_1 < x_2 < \cdots < x_n$. Then there is a cyclic permutation $\pi = \pi(R, F)$ associated to R, defined by $F(x_i) = x_{\pi(i)}$. A relation $\leq_1$ on the set C of all cyclic permutations can be constructed as follows: if $\sigma, \tau \in C$, then $\tau \leq_1 \sigma$ if and only if every map $F : \mathbf{R} \to \mathbf{R}$ which has a periodic orbit R with $\pi(R, F) = \sigma$ also has a periodic orbit S with $\pi(S, F) = \tau$. If this is the case then we shall say that the permutation σ *forces* the permutation τ. The relation $\leq_1$ is a partial order which can be calculated, in the sense that there is a simple algorithm to determine whether or not $\tau \leq_1 \sigma$ for any two cyclic permutations τ and σ: unfortunately this algorithm takes a long time to execute for orbits of high period, and it is difficult to use it to analyze the global structure of the partially ordered set $(C, \leq_1)$ [6]. There is, however, an interesting subset of C onto which the restriction of $\leq_1$ is well-understood. Let UM $\subseteq C$ be the set of *unimodal permutations*: that is, the set of permutations of periodic orbits of the *tent map* (or one-dimensional horseshoe)

$$F(x) = \begin{cases} 3x & 0 \leq x \leq 1/2 \\ 3(1-x) & 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(equivalently, UM is the set of permutations which can be realized by periodic orbits of unimodal maps whose turning point is a maximum). The partial order $\leq_1$ restricts to a linear order on UM which is described by kneading theory [19].

This is a statement of topological universality in the one-dimensional context: there is a universal order in which the periodic orbits are built up in families of unimodal maps of the line. In Appendix A we provide a listing of the unimodal order for orbits with periods 1 through 10. It is important to understand that the statement '(UM, $\leq$) is a linearly ordered set' gives constraints on the possible periodic orbit structure of all maps of the line, not just unimodal ones: similarly, the results which we shall present here are applicable to arbitrary orientation-preserving homeomorphisms of the plane, not just to those which are horseshoe-like.

A periodic orbit R of a homeomorphism of the plane can be described by its *braid type* [2]; the reader who lacks the mathematical background necessary to appreciate the following definitions should simply regard the braid type as a two-dimensional analog of the permutation, and the term 'pseudo-Anosov' as one which, while essential for the accurate statement of theorem 1, does not appear in later statements. Very roughly, braids can be divided into three types, the so-called *finite order braids* — braids which only force the existence of a finite number of other braids (periodic orbits), — *pseudo-Anosov braids* — braids which must force the existence of an infinite number of other braids, and *reducible braids* — braids which can be decomposed into distinct components which fall into one of the first two types. Pseudo-Anosov braids play an important role in the study of chaotic three-dimensional flows since, at the topological level, they force the coexistence of an infinite number of other periodic orbits. The term pseudo-Anosov arises within the context of the Nielsen-Thurston classification of isotopy classes of surface automorphisms, which is the main tool used in the proofs of the results presented here: the reader interested in this powerful theory could consult Ref. [20], in which there is also an extensive bibliography. An exposition of the relevance of Nielsen-Thurston theory to two-dimensional dynamics, with particular reference to braid types, can be found in Ref. [2].

Let f and g be orientation-preserving homeomorphisms of the plane which have periodic orbits R and S respectively. We say that (R, f) and (S, g) *have the same braid type* if there is an orientation-preserving homeomorphism $h : \mathbf{R}^2 \setminus R \to \mathbf{R}^2 \setminus S$ such that $h \circ f|_{\mathbf{R}^2 \setminus R} \circ h^{-1}$ is isotopic to $g|_{\mathbf{R}^2 \setminus S}$. Having the same braid type is an equivalence relation on the set of pairs (R, f), and we write $\text{bt}(R, f)$ for the equivalence class containing (R, f), the *braid type* of the periodic orbit R of f. We say that the braid type $\text{bt}(R, f)$ is of pseudo-Anosov, reducible, or finite order type, according to the Nielsen-Thurston type of the isotopy class of $f : (S^2, R \bigcup \{\infty\}) \to (S^2, R \bigcup \{\infty\})$ (see Ref. [21] for a statement of Thurston's classification theorem for homeomorphisms relative to a finite invariant set). By analogy with the one-dimensional theory, we define a relation $\leq_2$ on the set BT of all braid types as follows: if $\beta, \gamma \in$ BT, then $\gamma \leq_2 \beta$ if and only if every homeomorphism of the plane which has a periodic orbit of braid type β also has a periodic orbit of braid type γ. If this is the case, we shall say that the braid type β *forces* the braid type γ. The relation $\leq_2$ is a partial order [22]: as in the one-dimensional case, there is an algorithm for calculating this order (the so-called "train track algorithm") [7], but it is difficult to use it to analyze the global structure of the partially ordered set (BT, $\leq_2$). Still, using a version of this algorithm due to Bestvina and Handel [7], it is possible to calculate the orbits forced by an individual braid up to say period 10 or so by hand, and we have done this for several low period pseudo-Anosov horseshoes braids in order to find their associated topological entropies (see Table I). Also, an automated version of this algorithm promises to extend these calculations to higher period orbits [23].

We restrict our attention to the subset HS $\subseteq$ BT of *horseshoe braid types*: that is, those which are realized by periodic orbits of the horseshoe map f [24]. The determination of constraints on the order in which periodic orbits can be built up in the creation of horseshoes is tantamount to the analysis of the partially ordered set (HS, $\leq_2$); and an important observation in discussing the topological universality of two-dimensional systems is that, unlike (UM, $\leq_1$), the partially ordered set (HS, $\leq_2$) *is not linearly ordered.*

The connection between the ordered sets (UM, $\leq_1$) and (HS, $\leq_2$) is that periodic orbits of both the tent map and the horseshoe have a symbolic description. We associate to points in the non-wandering set of each map an infinite sequence of $0's$ and $1's$ in the usual manner, and describe a period n orbit R by its *code* c_R, given by the first n symbols of the sequence associated to the rightmost point of R. We shall identify periodic orbits of the two maps which have the same code, using the same symbol R to denote each orbit. Such an identified pair R will be referred to as a *horseshoe orbit* (or simply as an *orbit*): the fact that the orbit is periodic will always be assumed. We shall write c_R^∞ for the semi-infinite sequence $c_Rc_Rc_R\ldots$ (which is associated to the rightmost point of R, regarded as a periodic orbit of the tent map); and ${}^\infty c_R^\infty$ for the bi-infinite sequence $\ldots c_Rc_Rc_R\ldots$ (which is associated to the rightmost point R, regarded as a periodic orbit of the horseshoe).

It will be convenient for us to make another identification. If R is a period n orbit, then denote by $\tilde{c}_R$ the code obtained by changing the last symbol of c_R. If $\tilde{c}_R$ is not the two-fold repetition of a sequence of length $n/2$, then it is the code of a period n orbit R' which can be shown to have the same permutation and braid type as R. We identify R and R' in what follows, using the same symbol R to denote both, referring to them as a single orbit. The code c_R of the identified pair will always be chosen to end with a zero (this is to ensure that the sequence c_R^∞ contains the group of symbols $\ldots 010\ldots$, which is necessary for the algorithms which follow). After making this identification, distinct horseshoe orbits have distinct permutations: however, it is possible for several different orbits to have the same braid type. With this in mind, we say that a function defined on the set of all horseshoe orbits is a *braid type invariant* if it takes the same value on any two orbits of the same braid type.

Given two horseshoe orbits R and S, we shall write $S \leq_1 R$ if and only if $\pi(S,F) \leq_1 \pi(R,F)$; and we shall write $S \leq_2 R$ if and only if $\mathrm{bt}(S,f) \leq_2 \mathrm{bt}(R,f)$. We refer to $\leq_1$ and $\leq_2$ as the one- and two-dimensional *forcing orders* respectively. Our aim is to analyze $\leq_2$ using the well-understood properties of $\leq_1$. Notice that the orders $\leq_1$ and $\leq_2$ cannot strictly be regarded as being defined on the same set: if R and S are two horseshoe orbits which have the same braid type but different permutations, then they must be regarded as being equal when considering the order $\leq_2$, but as being distinct when considering the orbit $\leq_1$. For example, the orbits $f_{3\times 2}$ and s_6^2 in Table I are two such orbits (for an example analogous example in the orientation-reversing Henon map see Ref. [25]).

We say that a horseshoe orbit R is *quasi-one-dimensional* (qod) if for all orbits S we have $R \geq_1 S \Longrightarrow R \geq_2 S$. As proved in Ref. [17], the following result provides a simple classification of the qod orbits which have pseudo-Anosov braid type. Let $\hat{\mathbf{Q}}$ denote the set $\mathbf{Q}\cap(0,1/2)$ of rationals lying (strictly) between 0 and 1/2. Given such a rational $q = m/n$ (in lowest terms), we write P_q for the period $n+2$ orbit which has code $c_q = 10^{\kappa_1(q)}1^20^{\kappa_2(q)}1^2\ldots 1^20^{\kappa_m(q)}10$, where $\kappa_1(q) = \lfloor 1/q\rfloor - 1$, and $\kappa_i(q) = \lfloor i/q\rfloor - \lfloor (i-1)/q\rfloor - 2$ for $2 \leq i \leq m$ (here $\lfloor x\rfloor$ denotes the greatest integer which does not exceed x). Thus, for example, $c_{2/5} = 1011010$. Then we have:

Theorem 1 $q \longleftrightarrow P_q$ is a one-to-one correspondence between $\hat{\mathbf{Q}}$ and the set of quasi-one-dimensional horseshoe orbits which have pseudo-Anosov braid type. Moreover $P_q \geq_2 P_{q'} \iff q' \geq q$ for all $q, q' \in \hat{\mathbf{Q}}$.

It is a consequence of quasi-one-dimensionality that $P_q \geq_2 P_{q'} \iff P_q \geq_1 P_{q'}$, and therefore the braid types of the orbits P_q form a linearly ordered subset of BT. The final statement of the theorem tells us that the ordering within this subset is simply the reverse of the usual ordering on $\hat{\mathbf{Q}}$.

Theorem 1 allows us to quickly identify a very useful subset of pseudo-Anosov braids. Some of the low period qod (pseudo-Anosov) orbits are listed in Table II, and a program is easily written to quickly generate all the horseshoe qod orbits of any desired period. Moreover, Theorem 1 can be used to define an invariant of horseshoe braid type: the *height* $q(R)$ of a horseshoe orbit R is the unique element of $[0,1/2]$ with the property that for all $q \in \hat{\mathbf{Q}}$ we have $q < q(R) \Longrightarrow P_q \geq_1 R$ and $q > q(R) \Longrightarrow P_q \not\geq_1 R$ (thus $q(R) = \sup\{q \in \hat{\mathbf{Q}} : P_q \geq_1 R\}$). Because the orbits P_q are qod, it follows immediately that $q < q(R) \Longrightarrow P_q \geq_2 R$. In fact, it can be shown that the second property in the definition of height also holds for the two-dimensional forcing order: that is,

Theorem 2 Let R be a horseshoe orbit and $q \in \hat{\mathbf{Q}}$. Then $q < q(R) \Longrightarrow P_q \geq_2 R$ and $q > q(R) \Longrightarrow P_q \not\geq_2 R$. In particular, height is a braid type invariant.

We can also easily find the finite order braids because the finite order braids of the horseshoe are exactly those whose period equals the denominator of the height [17,26]. Because the height is defined in terms of the one-dimensional order, it can easily be calculated using kneading theory. We first define the height $q(c) \in (0,1/2]$ of a semi-infinite sequence which begins $c = 10\ldots$, and which contains the group of symbols $\ldots 010\ldots$. To do this, write c in the form $c = 10^{\kappa_1}1^{\mu_1}0^{\kappa_2}1^{\mu_2}\ldots$, where $\kappa_i \geq 0$ and μ_i is either 1 or 2 for each i, with $\mu_i = 1$ only if $\kappa_{i+1} > 0$ (thus the κ_i and μ_i are uniquely determined by c, and the fact that c contains the group $\ldots 010\ldots$ means that $\mu_s = 1$ for some s). Let

$$I_r(c) = \left(\frac{r}{2r + \sum_{i=1}^r \kappa_i}, \frac{r}{(2r-1) + \sum_{i=1}^r \kappa_i}\right]$$

for each $r \geq 1$, and let s be the least positive integer such that either $\mu_s = 1$ or $\cap_{i=1}^{s+1} I_i(c) = 0$. Then $\cap_{i=1}^s I_i(c) = (x,y]$ for some x and y: we define $q(c) = y$ if $\mu_s = 1$ or $I_{s+1}(c) > \cap_{i=1}^s(c)$, and $q(c) = x$ if $\mu_s = 2$ and $I_{s+1}(c) < \cap_{i=1}^s I_i(c)$.

Theorem 3 $q(R) = q(c_R^\infty)$: in particular, $q(R)$ is positive and rational.

For example, let R be the period 20 orbit with code $c_R = 10000110001100001100$. We have $\kappa_1 = 4$, $\mu_1 = 2$, $\kappa_2 = 3$, $\mu_2 = 2$, $\kappa_3 = 4$, $\mu_3 = 2$, $\kappa_4 = 2$, and $\mu_4 = 1$. Thus

$I_1 = (1/6, 1/5]$, $I_2 = (2/11, 2/10]$, $I_3 = (3/17, 3/16]$, and $I_4 = (4/21, 4/20]$. Since $4/21 > 3/16$ we have $I_1 \cap I_2 \cap I_3 \cap I_4 = 0$, and $I_4 > I_1 \cap I_2 \cap I_3$. Therefore $q(R) = \max(I_1 \cap I_2 \cap I_3) = 3/16$.

There is a relationship between height and a well-established braid type invariant: the height of a horseshoe is equal to the left hand endpoint of its rotation interval. A proof of this is given in Ref. [17], where a practical algorithm for determining the rotation interval of a horseshoe is described. It can also be shown using the algorithm for determining the height that for each $q = m/n$ (in lowest terms), P_q is the only period $n+2$ orbit of height q; thus it is the only orbit of its braid type. This observation enables us to determine exactly which of the orbits P_q are forced by a given orbit R in the two-dimensional order: we now present an algorithm for this purpose. If R is a horseshoe orbit then we define the *depth* $r(R) \in (0, 1/2] \cap \mathbf{Q}$ of R as follows: consider all groups of the form $\ldots 01110 \ldots$ or $\ldots 01010 \ldots$ in the sequence ${}^{\infty}c_R^{\infty}$; suppose that there are l such groups $g_1, \ldots, g_l$ contained in one period of ${}^{\infty}c_R^{\infty}$. If $l = 0$ then $r(R) = 1/2$. Otherwise, for each $i \leq l$ let f_i be the code obtained by starting at the last 1 in g_i and moving forwards through ${}^{\infty}c_R^{\infty}$, and b_i be that obtained by starting at the first 1 and moving backwards. Then $r(R) = \min_{1 \leq i \leq l} \max(q(f_i), q(b_i))$.

Theorem 4 $r(R)$ is the unique element of $[0, 1/2]$ with the property that for all $q \in \hat{\mathbf{Q}}$ we have $r(R) < q \implies R \geq_2 P_q$ and $r(R) > q \implies R \not\geq_2 P_q$. In particular, depth is a braid type invariant.

The following corollaries from theorems 2 and 4 will be very useful in our braid analysis since they allow to us to apply one-dimensional theory to calculate two-dimensional orbit forcings.

Corollary 1 Let R and S be horseshoe orbits. If $r(R) < q(S)$ then $R \geq_2 S$. On the other hand, if $q(R) < q(S)$ and $r(S) < r(R)$ then $R \not\geq_2 S$ and $S \not\geq_2 R$: thus orbits of the braids types R and S can exist independently of each other.

In fact, a much stronger result can be proved: if $r(R) < q(S)$ then every homeomorphism of the plane which has a periodic orbit of braid type R has at least as many periodic orbits of the braid type S as does the horseshoe (the corollary only says that every such homeomorphism has at least one such orbit). Thus all of the periodic orbits of the braid type S must be created before any of the periodic orbits of braid type R in any family of homeomorphisms leading to the creation of a horseshoe. We can use the following corollary to locate some of these orbits.

Corollary 2 If a homeomorphism of the disc has a qod orbit P_q, and R is another orbit which is forced by P_q in the one-dimensional order (which means that its height is $\geq q$), then the homeomorphism must have at least one orbit of the braid type of R. If in addition the height of R is *strictly* greater than q, then the homeomorphism must have at least as many orbits of the braid type of R as does the horseshoe [27].

For example, if a period 7 orbit with the braid type of 1011010 (height 2/5) is extracted, then there must be at least one orbit of the braid type of 10110 (since this braid type has height 2/5), and at least as many of the braid type of 101110 as in the horseshoe (i.e., 2) need exist (since the height of these orbits is also 2/5). On the other hand, both of the orbits in the period 6 pair 10111_1^0 must exist since the height of these orbits are 1/2. A listing of the low period qod orbits and some of the low period orbits which they force is found in Table III.

Since the qod orbits essentially inherit the one-dimensional forcing order we can use one-dimensional methods to calculate their topological entropies. This information can be used to obtain lower bounds for the topological entropy of a partially-formed horseshoe. More precisely, given a horseshoe orbit R, let $h(R)$ denote the smallest possible entropy of a homeomorphism having an orbit of the same braid type as R. Now we know that for all $q > r(R)$ we have $R \geq_2 P_q$, and hence the topological entropy $h(R) \geq h(P_q)$. However, since P_q is a qod orbit, it can be shown that $h(P_q)$ is equal to the entropy of P_q regarded as an orbit in one dimension: this can be calculated using standard transition matrix techniques. A *Mathematica* program based on the method of Block et. al. [28] has been written which allows us to calculate the topological entropies of the qod orbits. Some of the results are listed in Table II. A graph of the function $(q, h(P_q))$ appears in Ref. [17]; it is monotonic and discontinuous everywhere. Thus, we can go directly from a qod orbit to a lower bound for the topological entropy. For example, the orbit R with code $c_R = 10011010$ has $r(R) = 1/3$: any partially formed horseshoe which includes an orbit of the braid type of R has entropy greater than $h(P_{2/5}) \approx 0.442$. By taking rationals closer and closer to 1/3 we get better and better estimates (e.g., 5/14 gives 0.481). We can also use the pseudo-Anosov orbits which are not qod to get estimates (e.g., the orbit s_8^4 mentioned above has $h \approx 0.498$), however in this instance we must compute the entropy from a train track and this can be a difficult calculation.

In Table I we collect together some useful facts about braids in the horseshoe up to period eight. All the reducible orbits up to period nine only have finite order components. This is because the lowest period with a pseudo-Anosov orbit is five, and numbers less than or equal to nine do not have proper factors greater than or equal to five. As mentioned previously, the topological entropy for the pseudo-Anosov orbits which are not quasi-one-dimensional is calculated by finding the "train track" using the method of Bestvina and Handel [7]. This method results not only in a topological entropy, but also an explicit Markov partition (a 'topological model') and associated Perron-Frobenius matrix which is useful for locating in phase space where the predicted periodic orbits are to be found.

III. ROSSLER BRAID ANALYSIS

A braid analysis of a low dimensional chaotic time series consists of four steps once an appropriate three-dimensional space is created [15]: (i) the periodic orbits are extracted by the method of close recurrence [29,30], (ii) the braid type of each periodic orbit is identified and the orbits are ordered by their two-dimensional forcing relationship [7,17] (iii) a subset of braids are selected which have maximal forcing and which force the orbits extracted in step (i), and (iv) if possible, an attempt is made to verify that some of the predicted orbits (not originally extracted in step (i)) are found in the system.

In practice, steps (i) and (ii) are greatly simplified if the *template* or *knot-holder* organizing the flow can be identified using the procedure described by Mindlin and co-workers [4,15,31,32]. Knowledge of the template helps in obtaining the symbolic names of the periodic orbits and in calculating the forcing relationship for the specific braids in that template. For instance, if the template is identified as a two-branch horseshoe knot holder (as are all the examples studied in this paper), then the theory of qod orbits of section II can be applied to simplify the analysis.

Although template identification is very valuable, it is not essential for a braid analysis. Nor is the symbolic identification of the extracted orbits. In the worst case a braid analysis does require that the the braid conjugacy class of each extracted periodic orbit is identified (see Elrifai and Morton [33], or Jaquemard [34] for algorithms), and that the minimal Markov model (a 'train track' in the language of Thurston) can be constructed for each braid (see Bestvina and Handel [7], Los [8], and Franks and Misiurewicz [9] for algorithms). Algorithms exist for both of these steps, although the most computationally efficient version of the braid conjugacy algorithm is probably not an effective solution beyond B_8.

To illustrate the braid analysis we consider a chaotic attractor of the Rossler equations,

$$\dot{x} = -(y+z)$$
$$\dot{y} = x + ey$$
$$\dot{z} = f + xz - \mu z$$

with $e = 0.17$, $f = 0.4$ and $\mu = 0.85$. The Rossler equations are integrated through 10^5 cycles and the return map is examined at the half plane $\Sigma = \{(x,y,z) : x < 0 \ \& \ y = 0\}$. As shown in Fig. 1, the template is easily identified as a horseshoe with zero global torsion. This template identification is verified by calculating the relative rotation rates and linking numbers of the extracted periodic orbits as described by Mindlin et. al. [15,29,31].

To extract the (surrogate) periodic orbits by the method of close recurrence we first convert the return map from the sequence of values (x_n, z_n) directly into the symbol sequence of $0's$ and $1's$. In this particular instance, since the map is close to one-dimensional, a good symbolic partition is obtained by examining the maximum value of the next return map formed from the projection on the x-coordinate — $(-x_n, -x_{n+1})$ — at the surface of section. Orbits passing to the left of the maximum are labeled zero, and those to the right are labeled one. Next we search this symbolic encoding for each and every periodic symbol string. Every time a periodic symbol string is found we calculate its recurrence and then save the instance of the orbit with the best recurrence. For instance, in searching for the period three orbit '100' we search the symbolic encoding of the return map for any instance of '100' and its cyclic permutations '010' and '001', and every time this symbol string is found, we next calculate its recurrence, which for this period three orbit is $\epsilon_{y=0} = (x_{n+3} - x_n)^2 + (z_{n+3} - z_n)^2$, and then save the orbit with the minimum ϵ. The advantage of this procedure of orbit extraction is that it is exhaustive. We search for every possible orbit up to a given period. In these studies we searched for all orbits between periods 1 and 16.

The resulting spectrum of periodic orbits up to period eight is shown in Table IV. This should be compared with the Table in Appendix A. The orbits which are present in (the full shift) Table of Appendix A, and not present in Table IV, are said to be *pruned*. Our goal is to predict as best as possible the pruned spectrum from the chaotic time series.

Before we discuss the braid analysis, though, it is interesting to consider the number of orbits extracted as a function of the number of points in the return map. This is shown in Table V. As expected, the number of orbits that can be extracted increases with the number of points in the return map. More importantly, Table V strongly suggests that using the method of close recurrence it is possible to obtain all the low period periodic orbits embedded within the strange attractor. For instance, after 10^4 points are examined, we see that no new periodic orbits are found below period six. Similarly, after examining 10^5 points we believe we have found all orbits up to period eight. These results also caution us when working with small data sets — the extracted orbit spectrum is expected to miss orbits either because the orbit is pruned (it is not in the strange set) or because the sample of the strange set we are examining fails in providing a close enough coverage over the whole attractor.

Using the results in section II, the finite-order and quasi-one-dimensional orbits in the extracted spectrum are easily identified from their orbit codes (see Table IV). Not unexpectedly, we see a sequence of quasi-one-dimensional orbits of increasing entropy (decreasing height) — the maximal qod orbit is the period 16 orbit 1011011011011010 with entropy $h \approx 0.480804$. All orbits forced by this period 16 orbit up to period 8 are present, and none are missing. So this period 16 qod orbit already gives us a very good hyperbolic set with which to approximate to our (possibly nonhyperbolic) chaotic attractor. Can we do better? Doing better in this instance means identifying a pseudo-Anosov orbit which is not qod, but perhaps implies the maximal qod orbit found. Indeed, in this data set there is such an orbit, it is the period 8 orbit 10010100 with entropy $h \approx 0.498093$. Again, the spectrum of orbits forced by this maximal pA orbit are consistent with the extracted spectrum which was examined up to period 16.

This data set is close to one-dimensional so kneading theory also does quite well for predicting the low period orbits. For instance, we could consider the period 3 orbit 100, and this orbit (based on one-dimensional unimodal theory) also accurately predicts most of the extracted spectrum. However, this period 3 orbit is finite order, and we know from the results of Holmes and Whitley that there will be many (possibly high period) orbits which are forced by $\leq_1$ but not by $\leq_2$ [35] (in fact, we know that in 2-dimensions 100 forces only itself and a fixed point, and in 1-dimension it forces orbits of arbitrarily high period). Thus, although one-dimensional theory is a useful guide in this instance to the low period orbits, it can not be safely applied to make predictions about high period orbits.

IV. BELOUSOV-ZHABOTINSKII REACTION BRAID ANALYSIS

To further illustrate the braid analysis method we obtained data from the Belousov-Zhabotinskii chemical reaction [36]. This is the same data set analyzed by Mindlin et. al. and consists of 65,000 equally spaced points which measure the time dependence of the bromide ion concentration in the stirred chemical reactor. Following the techniques described by Mindlin et. al. we also embedded the scalar time series $x(i),\ i = 1, 2, \ldots, N$ in $\mathbf{R}^3$ via a *differential phase space* embedding described by

$$\begin{aligned} y_1(i) &= x(i) + \lambda * y_1(i-1), \quad \lambda = 0.995 \\ y_2(i) &= x(i) \\ y_3(i) &= x(i) - x(i-1) \end{aligned}$$

from which we reproduced the 3-dimensional attractor and return map shown in Figs. 5 and 6(a) of Ref. [15]. The attractor is a zero global torsion horseshoe. There are approximately 125 points per cycle so the return map consists of about 520 points.

Our technique for extracting (surrogate) periodic orbits from this time series differs somewhat from that described in Ref. [15]. We use the same procedure described for the Rossler data: first the return map data is converted into a symbol sequence of $0's$ and $1's$ depending on whether the orbit passes to the left or right of the maximum value of the return map, and second an exhaustive search is performed for all possible periodic orbits between periods 1 and 15. Again, this data set is almost one-dimensional and the simple symbolic prescription just described leads to a unique and consistent encoding of all the periodic orbits we are able to extract. Since the data set (at the return map) is small, we choose not to pick an arbitrary cut off for ϵ, the close return criterion. Rather, we report the best ϵ we are able to extract for a given periodic orbit (see Table VI). By including this additional piece of information we can make a more selective judgement about which close returns are, and are not, good surrogates for periodic orbits. In this way we are able to locate a few more periodic orbits than were originally reported in Ref. [15]. Also, one perhaps surprising result comes out of this extraction method. It is not uncommon to find orbits of high period with very small recurrences. For example the period 13 orbit 1011011101110 has a recurrence of $\epsilon = 0.000712$ which is significantly better than almost all of the lower period orbits.

A list of the extracted orbits and their Thurston types is presented in Table VI. As expected, there is a sequence of quasi-one-dimensional orbits of increasing entropy, the largest of which is the period 16 orbit 1011011011011011 with entropy $h \approx 0.48084$. In this particular instance, the period 16 qod orbit is in fact the maximal pseudo-Anosov orbit in the data set and it forces all the extracted orbits in this data set except for the finite order period three orbit 101. Indeed, a careful analysis of this data set suggests that the signal is subject to a small parametric drift which carries it between the strange attractor and a stable period 3 orbit (whose 1-dimensional entropy is $h \approx 0.4812$) [37].

Table VII shows the number of predicted and extracted orbits as a function of period. The number of forced orbits which are not in the extracted data set increases with the period. As with the Rossler data set, we believe the forced orbits which are missing could actually be extracted from the data set if we were given a longer time series which provides a better coverage of the entire attractor.

V. CONCLUSION

In retrospect, we find it remarkable that such a small subset of periodic orbits (which are rather easy to get from experiments) contain so much topological and dynamical information about a (low-dimensional) flow. As previously demonstrated, a few low period orbits are sufficient to determine the template describing the stretching and folding of the strange set [4,15,31]. The template provides an upper bound to the topological entropy and is, in a sense, a maximally (i.e., a full shift) hyperbolic set which we formally associate to a (possibly nonhyperbolic) strange set [29]. In this paper we show how a sequence of periodic orbits (and their associated hyperbolic sets) can be used to obtain a collection of finer and finer approximations to a strange set which is probably not hyperbolic. Formally, we might say that the hyperbolic set associated with each pseudo-Anosov braid is embedded within the strange attractor we are trying to describe in the sense that the (possibly nonhyperbolic) strange set must contain at least all the orbits forced by the extracted pseudo-Anosov braid. We indirectly discussed two measures of the goodness of this approximation — the difference in the topological entropy, and the difference between the forced and extracted low period orbits. Using either measure, we have seen that it is possible to select moderate period orbits (say < period 20) which provide good hyperbolic sets with which we can approximate a (possibly nonhyperbolic) strange set.

The dynamical information derived from an orbit depends only on its braid type. As mentioned in section III, the braid type of an orbit can be determined without obtaining a good symbolic description of the orbits in the flow. We will illustrate these techniques in a future paper in which we consider a braid analysis of the bouncing ball system in a low-dissipation regime [38]. Our strategy for handling the cases where a good symbolic description is not easy to obtain is to find a complete set of braid invariants on the small set of braids of interest. For instance, as mentioned in Table I, the exponent sum (simply the sum of the crossings) is a complete braid type invariant for horseshoe braids up to period 8.

We would also like to point out that it is common for a collection of finite order braids to be pseudo-Anosov. This suggests an alternative strategy: instead finding a single orbit with maximal implications, it should also be possible to find a collection of orbits (possibly with very low period) which force all the observed orbits and provides yet another hyperbolic approximating set. This is also the subject of future investigations.

ACKNOWLEDGMENTS

We wish to thank Bob Gilmore whose questions began this collaboration and Eric Kostelich for providing the Belousov-Zhabotinskii data file.

APPENDIX A: APPENDIXES

This Appendix shows the growth of periodic orbits in the unimodal map in Table VIII and the unimodal ordering in Table IX.

* Internet: nbt@reed.edu

[1] E. Artin, American Scientist **38**, 112 (1950).
[2] P. Boyland, "Braid types and a topological method of proving positive entropy" (1984, unpublished); Contemp. Math. **81**, 119 (1988).
[3] T. Matsuoka, in *Dynamical systems and nonlinear oscillations*, edited by G. Ikegami (World Scientific, Singapore, 1986) p. 58; Invent. Math. **70**, 319 (1983); J. Diff. Eqs. **76**, 190 (1988).
[4] P. Melvin and N. B. Tufillaro, Phys. Rev. A **44**, R3419 (1991), and references therein.
[5] A. Sarkovskii, Ukrain. Mat. Z. **16** (1964), 61; R. Devaney, *An introduction to Chaotic Dynamical Systems*, (Addison-Wesley, Reading, MA, 1989).
[6] S. Baldwin, Discrete Math. **67**, 111 (1987); C. Bernhardt, E. Coven, M. Misiurewicz, and I. Mulvey, Trans. Am. Math. Soc. **333**, 701 (1992).
[7] M. Bestvina and M. Handel, Ann. Math. **135**, 1 (1992); "Train tracks for surface homeomorphisms," (1992, unpublished).
[8] J. Los, "Pseudo-Anosov maps and invariant train tracks in the disc: a finite algorithm," Proc. Lond. Math. Soc. (1993, to appear).
[9] J. Franks and M. Misiurewicz, "Cycles for disk homeomorphisms and thick trees," (unpublished).
[10] D. Auerbach, P. Cvitanovic, J.-P. Eckmann, G. Gunaratne, and I. Procaccia, Phys. Rev. Lett. **58** 2387 (1987).
[11] N. B. Tufillaro, H. Solari, and R. Gilmore, Phys. Rev. A **41**, 5717 (1990).
[12] P. Cvitanovic and B. Eckhardt, Nonlinearity **6**, 277 (1993), and references therein.
[13] T. Li and J. Yorke, Amer. Math. Monthly **82**, 985 (1975).
[14] J. M. Gambaudo, S. van Strien, and C. Tresser, Ann. Inst. H. Poincare Phys. Theor. **50**, 335 (1989).
[15] G. B. Mindlin, H. Solari, M. Natiello, R. Gilmore, and X-J. Hou, J. Nonlinear Sci. **1**, 146 (1991); G. B. Mindlin and R. Gilmore, Physica D **58**, 229 (1992).
[16] P. Grassberger, H. Kantz, and U. Moenig, J. Phys. A **22**, 5217 (1989).
[17] T. D. H. Hall, "Weak universality in two-dimensional transitions to chaos," (1992, submitted to Phys. Rev. Lett.); "Periodicity in chaos: the dynamics of surface automorphisms," Ph.D. Thesis, University of Cambridge (1991, unpublished).
[18] M. Misiurewicz and Z. Nitecki, *Combinatorial Patterns for Maps of the Interval*, Mem. Am. Math. Soc. **456** (1991).
[19] J. Milnor and W. Thurston, in *Dynamical Systems*, edited by J. Alexander, Lecture Notes in Math. Vol. 1342 (Springer-Verlag, Berlin, 1988), p. 465.
[20] W. Thurston, Bull. Amer. Math. Soc. (N.S.) **19**, 417 (1988).
[21] M. Handel, Comm. Math. Phys. **127**, 339 (1990); R. C. Penner and J. L. Harer, *Combinatorics of Train Tracks*, Ann. Math. Studies **125** (Princeton Univ. Press, Princeton, 1992).
[22] P. Boyland, Comment. Math. Helv. (accepted for publication).
[23] T. White, (unpublished). A copy of the train track program is available by contacting T. White at "tadpole@ucrmath.ucr.edu".

[24] S. Smale, Bull. Amer. Math. Soc. **73**, 747 (1967).
[25] K. T. Hansen, Phys. Letts. A **165**, 100 (1992).
[26] P. Holmes and R. Williams, Arch. Rational Mech. Anal. **90**, 115 (1985).
[27] See Theorem 9.16 of Ref. [17].
[28] See Lemma 6.6 of Ref. [17]; L. Block, J. Guckenheimer, M. Misiurewics, and L-S. Young, in *Global Theory of dynamical Systems*, Lecture Notes in Mat. vol. 819 (Springer-Verlag, New York, 1980), p. 18; program is available from first author.
[29] N. B. Tufillaro, T. A. Abbott, and J. P. Reilly, *An experimental approach to nonlinear dynamics and chaos* (Addison-Wesley, Reading, MA, 1992).
[30] K. Pawelzik and H. G. Schuster, Phys. Rev. A **43**, 1808 (1991).
[31] G. B. Mindlin, X-J. Hou, H. G. Solari, R. Gilmore, and N. B. Tufillaro, Phys. Rev. Lett. **64**, 2350 (1990); F. Papoff, E. Arimondo, F. Fioretto, G. B. Mindlin, H. G. Solari, and R. Gilmore, Phys. Rev. Lett. **68**, 1128 (1992).
[32] N. B. Tufillaro, R. Holzner, L. Flepp, E. Brun, M. Finardi, and R. Badii, Phys. Rev. A **44**, R4786 (1991).
[33] E. A. Elrifai and H. R. Morton, "Algorithms for positive braids," (1990, unpublished).
[34] A. Jaquemard, J. Pure Appl. Algebra **63**, 161 (1990).
[35] P. Holmes and D. Whitley, Philos. Trans. Roy. Soc. London Ser. A **311**, 43 (1984).
[36] A. Arnedo, F. Argol, J. Elezgaray, and P. Richetti, Physica D **62**, 134 (1993); D. P. Lathrop and E. J. Kostelich, Phys. Rev. A **40**, 4028 (1989); K. Coffman, W. D. McCormick, A. Noszticzius, R. H. Simoyi, and H. Swinney, J. Chem. Phys **86**, 119 (1987); J. C. Roux, R. H. Simoyi, and H. L. Swinney, Physica D **8**, 257 (1983).
[37] R. Gilmore, private communication.
[38] N. B. Tufillaro and T. D. H. Hall, "Braid analysis of a bouncing ball," (in preparation).

FIG. 1. Orbits in the Rossler Attractor (e = 0.17, f = 0.4, μ = 8.5): (a) Strange attractor showing surface of section, (b) the 1 and 10 orbits extracted from the chaotic time series, (c) the pretzel knot 1011010 extracted from the chaotic time series, (d) the pretzel knot shown as a braid.

TABLE I. Low period horseshoe orbits. The exponent sum is a complete braid type invariant for all horseshoe orbits up to period eight (see Ref. [17] for the explicit conjugacy). The braid name notation is from Ref. [35]. Thurston braid types: finite order (fo), reducible (red), and pseudo-Anosov (pA), and quasi-one-dimensional (qod). The "red, fo" orbits are reducible Thurston types with finite order components.

P braid name	c_P symbolic name	π_P permutation	*Type* Thurston type	es exponent sum	$q(P)$ height	h topological entropy
s_1	1	(1)	fo	0	1/2	0
s_2	10	(12)	fo	1	1/2	0
s_3	10_1^0	(123)	fo	2	1/3	0
$f_{2\times 2}$	1011	(1324)	red, fo	5	1/2	0
s_4	100_1^0	(1234)	fo	3	1/4	0
s_5^1	1011_1^0	(13425)	fo	8	2/5	0
s_5^2	1001_1^0	(12435)	pA, qod	6	1/3	0.543535
s_5^3	1000_1^0	(12345)	fo	4	1/5	0
s_6^1	10111_1^0	(143526)	red, fo	13	1/2	0
$f_{3\times 2}$	100101	(135246)	red, fo	9	1/3	0
s_6^2	10011_1^0	(124536)	red, fo	9	1/3	0
s_6^3	10001_1^0	(123546)	pA, qod	7	1/4	0.632974
s_6^4	10000_1^0	(123456)	fo	5	1/6	0
s_7^1	101111_1^0	(1453627)	fo	18	3/7	0
s_7^2	101101_1^0	(1462537)	pA, qod	16	2/5	0.442138
s_7^3	100101_1^0	(1362547)	pA	14	1/3	0.476818
s_7^4	100111_1^0	(1254637)	pA	14	1/3	0.476818
s_7^5	100110_1^0	(1356247)	fo	12	2/7	0
s_7^6	100010_1^0	(1246357)	pA	10	1/4	0.382245
s_7^7	100011_1^0	(1235647)	pA	10	1/4	0.382245
s_7^8	100001_1^0	(1234657)	pA, qod	8	1/5	0.666213
s_7^9	100000_1^0	(1234567)	fo	6	1/7	0
$f_{2^2\times 2}$	10111010	(15472638)	red, fo	23	1/2	0
s_8^1	1011111_1^0	(15463728)	red, fo	25	1/2	0
s_8^2	1011011_1^0	(14725638)	fo	21	3/8	0
s_8^3	1001011_1^0	(13725648)	pA	19	1/3	0.346034
s_8^4	1001010_1^0	(13647258)	pA	17	1/3	0.498093
s_8^5	1001110_1^0	(13657248)	pA	17	1/3	0.498093
s_8^6	1001111_1^0	(12564738)	pA	19	1/3	0.346034
s_8^7	1001101_1^0	(12573648)	pA	17	1/3	0.498093
$f_{2\times 4}$	10001001	(13572468)	red, fo	13	1/4	0
s_8^8	1000101_1^0	(12473658)	pA	15	1/4	0.568666
s_8^9	1000111_1^0	(12365748)	pA	15	1/4	0.568666
s_8^{10}	1000110_1^0	(12467358)	red, fo	13	1/4	0
s_8^{11}	1000010_1^0	(12357468)	pA	11	1/5	0.458911
s_8^{12}	1000011_1^0	(12346758)	pA	11	1/5	0.458911
s_8^{13}	1000001_1^0	(12345768)	pA, qod	9	1/6	0.680477
s_8^{14}	1000000_1^0	(12345678)	fo	7	1/8	0

TABLE II. Topological entropy of *quasi-one-dimensional* (qod) orbits in a horseshoe up to period sixteen.

Period	Height	Name	Entropy
5	1/3	1001_1^0	0.543535
6	1/4	10001_1^0	0.632974
7	2/5	101101_1^0	0.442138
7	1/5	100001_1^0	0.666213
8	1/6	1000001_1^0	0.680477
9	3/7	10111101_1^0	0.397081
9	2/7	10011001_1^0	0.604904
9	1/7	10000001_1^0	0.687025
10	3/8	101101101_1^0	0.473404
10	1/8	100000001_1^0	0.690145
11	4/9	1011111101_1^0	0.373716
11	2/9	1000110001_1^0	0.65548
11	1/9	1000000001_1^0	0.691662
12	3/10	10011011001_1^0	0.600332
12	1/10	10000000001_1^0	0.692409
13	5/11	101111111101_1^0	0.361049
13	4/11	101101101101_1^0	0.479458
13	3/11	100110011001_1^0	0.609
13	2/11	100001100001_1^0	0.675816
13	1/11	100000000001_1^0	0.692779
14	5/12	1011110111101_1^0	0.41211
14	1/12	1000000000001_1^0	0.692964
15	6/13	10111111111101_1^0	0.354176
15	5/13	10110111101101_1^0	0.467734
15	4/13	10011011011001_1^0	0.599566
15	3/13	10001100110001_1^0	0.654699
15	2/13	10000011000001_1^0	0.684869
15	1/13	10000000000001_1^0	0.693056
16	5/14	101101101101101_1^0	0.480804
16	3/14	100011000110001_1^0	0.656227
16	1/14	100000000000001_1^0	0.693101

TABLE III. A sequence of *quasi-one-dimensional* (qod) orbits with increasing forcing implications (entropy). Thus, every orbited forced by the period 7 qod orbit is also forced by the period 10 orbit, and so on. The notation “(*)” after an orbit indicates that one (but perhaps not both) of the saddle-node pair is forced. Both saddle-node partners will be forced by the next orbit in the qod sequence.

Period	Forced by (7) 101101_1^0	Forced by (10) 101101101_1^0	Forced by (13) 101101101101_1^0
1			
2	10		
3			
4	1011		
5	1011_1^0(*)		
6	10111_1^0		
7	101111_1^0	101101_1^0	
8	10111010 1011111_1^0	1011011_1^0(*)	
9	10111111_1^0 10111101_1^0 10110101_1^0(*)	10110111_1^0	
10	101111111_1^0 101111101_1^0 101110101_1^0 101101111_1^0(*)	1011010111 101101101_1^0	101101101_1^0
11	1011111111_1^0 1011111101_1^0 1011110111_1^0 1011110101_1^0 1011011111_1^0(*) 1011011101_1^0(*)	1011010111_1^0 1011010101_1^0	1011011011_1^0(*)
⋮	⋮	⋮	⋮

TABLE IV. Periodic orbits extracted from time series data of the Rossler equations. All extracted orbits up to period 8 are shown. Between periods 9 and 16 only the extracted quasi-one-dimensional (*qod*) are shown. The maximal pseudo-Anosov orbit (1001010_1^0) is not qod, but it forces all qod orbits with height greater than 1/3. Thurston Types: finite order (fo), reducible (red), and pseudo-Anosov (pA).

Period	Name	Height	Entropy	Type
1	1	1/2	0	fo
2	10	1/3	0	fo
3	101	1/3	0	fo
3	100	1/3	0	fo
4	1011	1/2	0	red
5	10111	2/5	0	fo
5	10110	2/5	0	fo
6	101110	1/2	0	red
6	101111	1/2	0	red
6	100101	1/3	0	red
7	1011111	3/7	0	fo
7	1011110	3/7	0	fo
7	1011010	2/5	0.442138	pA, qod
7	1011011	2/5	0.442138	pA, qod
7	1001011	1/3	0.476818	pA
7	1001010	1/3	0.476818	pA
8	10111010	1/2	0	red
8	10111110	1/2	0	red
8	10111111	1/2	0	red
8	10110111	3/8	0	fo
8	10110110	3/8	0	fo
8	10010110	1/3	0.346034	pA
8	10010111	1/3	0.346034	pA
8	10010101	1/3	0.498093	pA
8	10010100	1/3	0.498093	pA
9	101111010	3/7	0.397081	pA, qod
9	101111011	3/7	0.397081	pA, qod
10	1011011010	3/8	0.473404	pA, qod
10	1011011011	3/8	0.473404	pA, qod
11	10111111010	4/9	0.373716	pA, qod
13	1011011011010	4/11	0.479450	pA, qod
13	1011011011011	4/11	0.479450	pA, qod
15	101111111111011	6/13	0.354176	pA, qod
15	101101111011010	5/13	0.467734	pA, qod
16	1011011011011010	5/14	0.480804	pA, qod

TABLE V. Number of periodic orbits extracted ($\epsilon = 0.005$) as a function of the number of points in the return map for time series data from the Rossler equation. The data suggests that after 100,000 points, all the periodic orbits in the time series up to period eight are extracted.

Period	100	500	1000	10,000	50,000	100,000
1	0	1	1	1	1	1
2	0	1	1	1	1	1
3	0	0	2	2	2	2
4	0	0	1	1	1	1
5	0	0	0	2	2	2
6	1	1	1	3	3	3
7	1	1	2	6	6	6
8	1	2	2	8	9	9
9	0	2	4	9	14	14
10	0	1	4	11	17	19

TABLE VI. Periodic orbits extracted from the Belousov-Zhabotinskii Reaction time-series up to period 15. All orbits with a best (normalized) recurrence of less than 0.1 are shown. The period 16 orbits are from Ref. [15].

Period	Name	Recurrence	Type
1	1	0.016782	fo
2	10	0.002615	fo
3	101	0.000128	fo
4	1011	0.002648	fo
5	10111	0.002962	fo
5	10110	0.013668	fo
6	101110	0.006449	fo
6	101111	0.029014	fo
7	1011110	0.005088	fo
7	1011010	0.041837	pA, qod, $h = 0.442138$
7	1011011	0.010585	pA, qod, $h = 0.442138$
8	10111010	0.014287	fo
8	10110111	0.017370	fo
8	10110110	0.070293	fo
9	101111010	0.001720	pA, qod, $h = 0.397081$
9	101101011	0.061843	pA
9	101101010	0.020884	pA
9	101101110	0.018380	pA
9	101101111	0.050094	pA
10	1011101010	0.037910	fo
10	1011101011	0.007237	fo
10	1011111010	0.035966	fo
10	1011011010	0.003440	pA, qod, $h = 0.473404$
10	1011011011	0.008971	pA, qod, $h = 0.473404$
11	10110101010	0.038044	
11	10110111010	0.009119	
11	10110110111	0.008447	
11	10110110110	0.085593	fo
12	101110101011	0.031644	
12	101110101010	0.013333	
12	101101010110	0.029674	
12	101101111010	0.055765	
12	101101101011	0.032222	
12	101101101010	0.016560	
12	101101101110	0.048563	
12	101101101111	0.064420	
13	1011110101010	0.052853	
13	1011010111010	0.022397	pA
13	1011010101110	0.004291	
13	1011011101110	0.000712	
13	1011011101010	0.011199	pA
13	1011011111010	0.046255	pA
13	1011011010110	0.097165	pA, qod, $h = 0.479450$
13	1011011011010	0.013063	pA, qod, $h = 0.479450$
13	1011011011011	0.004663	
14	10110101110110	0.056515	
14	10110101010110	0.009366	
14	10110111010110	0.047037	
14	10110110111010	0.060024	
14	10110110110110	0.004920	fo
15	101101011101110	0.034250	
15	101101110101010	0.015669	
15	101101101011010	0.010945	
15	101101101010110	0.076061	

15	101101101110110	0.049261	pA
15	101101101111010	0.007018	
15	101101101101010	0.049573	
15	101101101101110	0.010284	
15	101101101101111	0.078043	
16	1011011011101010	na	pA
16	1011011011111010	na	pA
16	1011011011011010	na	pA, qod, $h = 0.480804$
16	1011011011011011	na	pA, qod, $h = 0.480804$

TABLE VII. Number of periodic orbits extracted and predicted for time series data from the Belousov-Zhabotinskii reaction ($\approx$ 500 points in the return map). As illustrated with the Rossler data, we expect that the time series is far to short to be able to extract all the predicted orbits of any except the lowest periods.

Period	# predicted	# found	# missing	# marginal
1	1	1		
2	1	1		
3	1	1		
4	1	1		
5	2	2		
6	2	2		
7	4	3	1	
8	5	2	2	1
9	8	4	2	2
10	11	5	6	

TABLE VIII. The number of periodic orbits in a horseshoe of least period n is given given by $(\sum_d 2^d \phi(n/d))/n$ where d ranges over all the divisors of n and ϕ is the Mobius function defined by $\phi(m) = 0$ if m has a repeated prime factor, otherwise $\phi(m) = (-1)^{(number\ of\ prime\ factors\ of\ m)}$.

Period	Number of Periodic Orbits of Least Period
1	2
2	1
3	2
4	3
5	6
6	9
7	18
8	30
9	56
10	99
11	186
12	335
13	630
14	1161
15	2182
16	4080
17	7710
18	14532
19	27594
20	52377
21	99858
22	190557
23	364722
24	698870
25	1342176
26	2580795
27	4971008
28	9586395
29	18512790
30	35790267

TABLE IX. One-dimensional unimodal ordering up to period 10. Orbits are sorted first by period and second by unimodal ordering. Orbits agreeing in all but the last digit (0,1) are braid equivalent saddle-node pairs.

Period	Name
1	0
1	1
2	10
3	101
3	100
4	1011
4	1001
4	1000
5	10111
5	10110
5	10010
5	10011
5	10001
5	10000
6	101110
6	101111
6	100101
6	100111
6	100110
6	100010
6	100011
6	100001
6	100000
7	1011111
7	1011110
7	1011010
7	1011011
7	1001011
7	1001010
7	1001110
7	1001111
7	1001101
7	1001100
7	1000100
7	1000101
7	1000111
7	1000110
7	1000010
7	1000011
7	1000001
7	1000000
8	10111010
8	10111110
8	10111111
8	10110111
8	10110110
8	10010110
8	10010111
8	10010101
8	10010100
8	10011100
8	10011101
8	10011111
8	10011110
8	10011010
8	10011011
8	10001001
8	10001011
8	10001010
8	10001110
8	10001111
8	10001101
8	10001100
8	10000100
8	10000101
8	10000111
8	10000110
8	10000010
8	10000011
8	10000001
8	10000000
9	101111111
9	101111110
9	101111010
9	101111011
9	101101011
9	101101010
9	101101110
9	101101111
9	100101100
9	100101101
9	100101111
9	100101110
9	100101010
9	100101011
9	100111011
9	100111010
9	100111110
9	100111111
9	100111101
9	100111100
9	100110100
9	100110101
9	100110111
9	100110110
9	100110010
9	100110011
9	100010011
9	100010010
9	100010110
9	100010111
9	100010101
9	100010100
9	100011100
9	100011101
9	100011111
9	100011110
9	100011010
9	100011011
9	100011001
9	100011000
9	100001000
9	100001001
9	100001011
9	100001010

9		100001110
9		100001111
9		100001101
9		100001100
9		100000100
9		100000101
9		100000111
9		100000110
9		100000010
9		100000011
9		100000001
9		100000000
10		1011101010
10		1011101011
10		1011111011
10		1011111010
10		1011111110
10		1011111111
10		1011010111
10		1011011111
10		1011011110
10		1011011010
10		1011011011
10		1001011011
10		1001011010
10		1001011110
10		1001011111
10		1001011101
10		1001011100
10		1001010100
10		1001010101
10		1001010111
10		1001010110
10		1001110010
10		1001110110
10		1001110111
10		1001110101
10		1001110100
10		1001111100
10		1001111101
10		1001111111
10		1001111110
10		1001111010
10		1001111011
10		1001101011
10		1001101010
10		1001101110
10		1001101111
10		1001101101
10		1001101100
10		1001100100
10		1001100101
10		1001100111
10		1001100110
10		1000100110
10		1000100111
10		1000100101
10		1000100100
10		1000101100
10		1000101101
10		1000101111
10		1000101110
10		1000101010
10		1000101011
10		1000101001
10		1000101000
10		1000111000
10		1000111001
10		1000111011
10		1000111010
10		1000111110
10		1000111111
10		1000111101
10		1000111100
10		1000110100
10		1000110101
10		1000110111
10		1000110110
10		1000110010
10		1000110011
10		1000010001
10		1000010011
10		1000010010
10		1000010110
10		1000010111
10		1000010101
10		1000010100
10		1000011100
10		1000011101
10		1000011111
10		1000011110
10		1000011010
10		1000011011
10		1000011001
10		1000011000
10		1000001000
10		1000001001
10		1000001011
10		1000001010
10		1000001110
10		1000001111
10		1000001101
10		1000001100
10		1000000100
10		1000000101
10		1000000111
10		1000000110
10		1000000010
10		1000000011
10		1000000001
10		1000000000