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Template analysis for a chaotic NMR laser

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The template (or knot holder) is extracted from the low-order unstable periodic orbits in the experimental time series of a chaotic NMR laser. The analysis indicates that the topological organization of the chaotic motion is governed by a horseshoe with a global torsion of zero in the parameter region examined.

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Various techniques for the extraction of unstable periodic orbits from chaotic time series have been investigated recently [1]. The knowledge of these dynamical invariants allows one to construct hierarchical approximations to the underlying dynamics [2]. In particular, Mindlin *et al.* [3] describe how periodic orbits can provide an integer classification for the topological organization of a strange attractor. In this Rapid Communication we apply this integer classification to an experimentally recorded strange attractor arising from an NMR laser [4,5]. Our aim is twofold: first, we improve our understanding of chaos in the NMR laser and, second, we provide further evidence for the experimental feasibility of the integer classification of strange attractors.

Our analysis closely follows the pioneering work of Mindlin *et al.* in their study of the Belousov-Zhabotinskii reaction [6]. The procedure consists of three steps, once the periodic orbits are extracted by the method of close recurrence [5]: first, a symbolic encoding of the periodic points is determined from an approximation to the generating partition of the return map; second, the relative rotation rates of the periodic orbits are calculated in a suitable embedding space [6–8]; and, third, the template (or knot holder) is identified from the linking numbers (and local torsion) of a few of the low-order periodic orbits [3,6]. The template extracted from the experimental time series provides a good first basis for the development of a more complete model of the dynamics since it generates integer invariants that any proposed model of the experimental system must satisfy.

Chaotic behavior has been studied in the NMR laser for over a decade [4]. Chaotic motions can occur in the NMR laser when the cavity quality factor $Q(t)$ is sinusoidally modulated, i.e., $Q(t)=Q_0(1+A_0\cos\Omega t)$, where $\Omega=2\pi\times 120$ Hz. The fact that the system is externally forced considerably simplifies the nonlinear analysis. The chaotic signal is recorded as a scalar time series $\{\xi_1,\xi_2,\ldots,\xi_N\}$, consisting of $N=8\times 10^5$ 12-bit integers, by sampling the transverse magnetization $M_t(t)$ with a frequency $\nu=25/T$, where $T=2\pi/\Omega$ is the period of the forcing term, i.e., $\xi_i=M_t(i/\nu)$. The data are then embedded in an E-dimensional space by constructing vectors of the form $\mathbf{v}_k=\{\xi_k,\xi_{k+5},\ldots,\xi_{k+5(E-1)}\}$, where $5/\nu$ is the appropriate delay time. The data are low noise and drift-free. All periodic orbits up to order 9 have been extracted by the method of close recurrence [5]. Our template analysis begins with these periodic orbits.

The symbolic name for each periodic point is obtained from the first return map ξ_{i+25} vs ξ_i. Although the resulting plot, displayed in Fig. 1, is not strictly one dimensional, it is close enough to a one-humped map of the interval to suggest connecting the maximum of each leaf of the attractor in order to obtain a sensible prescription for the symbolic encoding. This partition is also illustrated in

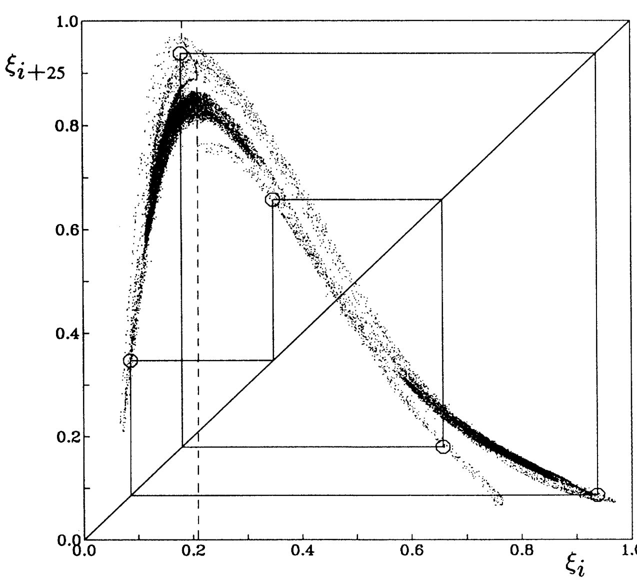

FIG. 1. First return map of the chaotic NMR laser time series showing the approximate partition for the symbolic encoding of the periodic orbits. A period-5 cycle is also displayed.

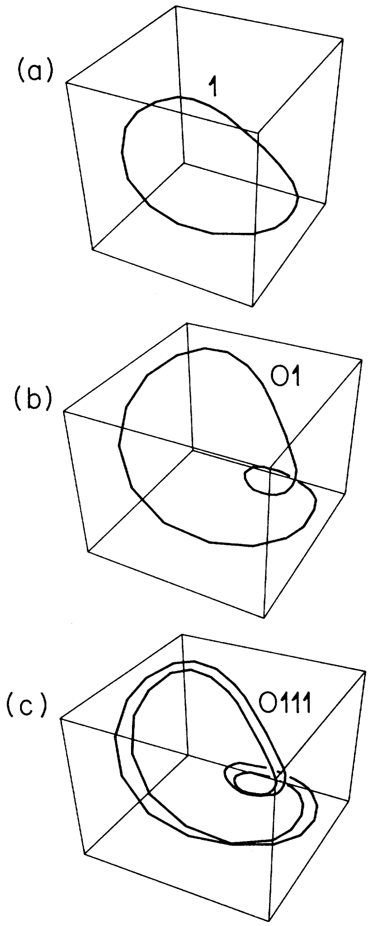

FIG. 2. Low-order periodic orbits extracted from the chaotic time series in the embedding space $(\theta(i),\xi_i,\xi_{i+5})$: (a) period-1 orbit with symbolic name 1, (b) the 01 period-2 orbit, and (c) the 0111 period-4 orbit.

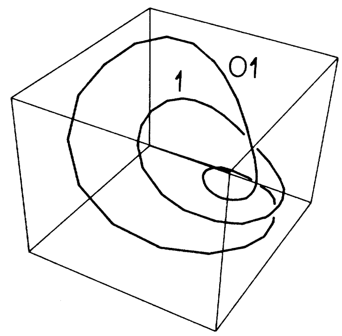

FIG. 3. The linking of the 1 and 01 orbits in the embedded phase space. The period-1 and period-2 orbits have linking number 1, and a relative rotation rate of $\frac{1}{2}$.

Fig. 1 [9]. All points which lie to the left of the partitioning curve are on the orientation preserving branch of the map and are assigned the symbolic name 0; all those on the orientation reversing branch are assigned the symbolic name 1. Each period-nT orbit of the flow corresponds to a collection of n points on the return map and is associated with a finite sequence of 0's and 1's which we take as its symbolic encoding. All symbolic names are equivalent up to cyclic permutations. To period 6, the periodic orbits have the following symbolic names: 1, 01, 011, 0111, 01111, 01101, 011111, 011101. The only forbidden sequence (to this degree of resolution) is 00. We next consider a three-dimensional embedding of both scalar time series and unstable periodic orbits. The three embedding variables for our analysis are the phase $\theta(i)=2\pi \times (1 \bmod 25)$ of the forcing term, the original scalar times series $\{\ldots,\xi_i,\ldots\}$ and the delayed one $\{\ldots,\xi_{i+5},\ldots\}$. The topology of the phase space is hence $S^1\times\mathbb{R}^2$. Figure 2 shows the knots formed by three low-order periodic orbits in this embedding space. A simple inspection of such three-dimensional plots often reveals the parentage of a periodic orbit. For instance, the 0111 orbit [Fig. 2(c)] is clearly the period-doubled progeny of the 01 orbit [Fig. 2(b)]. Additionally, superposition of three-dimensional orbit plots allows us to check visually the linking numbers and relative rotation rates of orbit pairs. For instance, Fig. 3 shows that the 1 and 01 orbits are linked once.

As argued by Mindlin *et al.* [6], the three lowest-order periodic orbits (1, 01, 011) provide sufficient information to calculate the form of the template. From two-data segments of the period-1 orbit we calculate a local torsion of 1 for the orientation reversing branch of the template. The linking numbers of the 1, 01, and 011 orbits are also calculated from their relative rotation rates and are iden-

TABLE I. Intertwining matrix for the chaotic NMR laser up to period 6. Experimental values are indicated in parentheses where they disagree from the relative rotation rates of a horseshoe.

	1	01	011	0111	01111	01101	011111	011101
1	0							
01	$\frac{1}{2}$	$0, \frac{1}{2}$						
011	$\frac{1}{3}$	$\frac{1}{3}$	$0, \frac{1}{3}$					
0111	$\frac{1}{2}$	$\frac{1}{4}, \frac{1}{2}$ $(\frac{1}{2})$	$\frac{1}{3}$	$0, \frac{1}{4}, \frac{1}{2}$				
01111	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{3}$ $(\frac{2}{5})$	$\frac{2}{5}$	$0, \frac{2}{5}$			
01101	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{2}{5}$	$0, \frac{2}{5}$		
011111	$\frac{1}{2}$ $(\frac{2}{3})$	$\frac{1}{3}, \frac{1}{2}$ $(\frac{1}{2})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{2}{5}$	$0, \frac{1}{3}, \frac{1}{2}$	
011101	$\frac{1}{2}$	$\frac{1}{3}, \frac{1}{2}, (\frac{1}{2}, \frac{2}{3})$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{3}, \frac{1}{2}$	$0, \frac{1}{3}, \frac{1}{2}$

tical to those discovered by Mindlin *et al.* for the Belousov-Zhabotinskii reaction. Therefore, following the arguments of Ref. [6], we predict that the topological organization of the NMR laser is governed by a horseshoe with a global torsion of zero.

In order to confirm (or invalidate) the horseshoe model we next calculate the relative rotation rates of all orbits up to period 6 from the experimental data [8]. The results are tabulated in the intertwining matrix shown in Table I. Whenever theoretical and experimental values do not coincide, we report the latter ones in parentheses.

Out of the first 36 relative rotation rates calculated, Table I shows four discrepancies between theoretical values predicted by a horseshoe model [7,8] and the experimental values measured in the NMR laser. At present, we believe that these discrepancies are the result of spurious crossings (noncrossings) that can arise in the experimental time series due to the discrete sampling and to noise. We are currently working to improve our embedding procedure in order to optimize the spatial separation of periodic orbits to resolve this difficulty.

In conclusion, we calculated the relative rotation rates from an experimental chaotic time series of the NMR laser. This analysis indicates that the topology organizing the strange attractor is governed by a horseshoe with global torsion zero in the parameter regime investigated. Further, we believe that the wealth of clean data available from the NMR laser makes this an excellent system for future investigations into the topological characterization of chaos.

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[8] N. B. Tufillaro, J. P. Reilly, and T. A. Abbott, *An Experimental Approach to Nonlinear Dynamics and Chaos* (Addison-Wesley, New York, 1991). See Appendices F and G for computer codes to calculate relative rotation rates.

[9] This partition is very close to the preimage of the one obtained in [5] directly for the Poincaré map. Both of them assign an unique symbolic encoding to all periodic points up to period 9.