

Templates and framed braids

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We show how a template for a dynamical system can be uniquely specified by a framed braid. This leads to a homological classification of strange attractors in terms of an associated linking matrix.

A recent Letter by Mindlin *et al.* [1] describes how periodic orbits embedded in a three-dimensional flow can be used to provide an integer characterization of a strange set. The central object in this topological theory of low-dimensional chaos is the embedded *template* (or *knot holder*), introduced in the general context of hyperbolic flows by Birman and Williams [2] (see also [3,4]). The template is a branched surface with a semiflow having the same spectrum of periodic orbits (up to isotopy) with the same topological organization as the original flow. Mindlin *et al.* [1,5] give a partial algebraic characterization of a k -branch template (and its associated strange attractor) in terms of two pieces of data: a $k \times k$ *template matrix*, and a $1 \times k$ *layering array*. In particular, these data determine the linking numbers of all pairs of periodic orbits in the flow, as well as the more delicate relative rotation rates defined by Solari and Gilmore [6]. Since periodic orbits are dense in a hyperbolic strange set, this provides a type of homological classification of the strange attractor of the flow. In this Rapid Communication, we propose an alternative (and equivalent) classification in terms of a single $k \times k$ *framed-braid linking matrix*. Our formulation arises from a canonical correspondence between templates and framed braids, which we describe first.

Recall that a *framed braid* is a geometric braid [7] with an integer associated to each strand. This integer, called the *framing* or *local torsion*, represents an internal structure of the strand. More precisely, it represents an isotopy class of framings of the normal bundle of the strand (fixed on the boundary) by giving the winding number (in multiples of π) of the normal frame about the strand.

A framed braid can be represented geometrically by a *ribbon graph*, obtained from the braid by replacing each strand by a ribbon. Each ribbon is given the integer number of half-twists corresponding to its framing, using the standard crossing conventions illustrated in Fig. 1 (which are opposite to those of Artin [7]). Algebraically, a framed braid can be specified by a *braid word* (unique up to equivalence in Artin's braid group [7]) together with the *framing* (a list of integers giving the framings of the strands, from left to right at the top). For example, the ribbon graph shown in Fig. 2 is specified by the braid

$\sigma_2 \sigma_3^{-1} \sigma_1$ with framing $(0,1,1,-2)$.

A closely related notion is that of a *layering graph*, made up of a *joining* part followed by a *splitting* part. The joining part consists of a collection of ribbons which descend from a fixed ordered position (as in a ribbon graph, but without twisting or intertwining) to a *branch line* where they are glued together into one ribbon. In the splitting part, this ribbon continues to descend, splitting back up into the original number of ribbons which return (again without twisting or intertwining) to their original positions. Labeling the ribbons $1, \dots, k$ (from left to right), a layering graph can be specified algebraically by a list of integers, the *layering array*, giving the order in which the ribbons are glued at the branch line (from back to front). Thus the first integer gives the label of the backmost ribbon, and so forth. The layering graph corresponding to the list $(1, \dots, k)$ will be called the *standard layering graph*. Figure 3 shows two layering graphs; the second one is standard.

Now we return to templates. Our discussion will be limited to *simple templates*, that is, templates in \mathbb{R}^3 for which all the branches are joined at the same branch line. By a result of Franks and Williams [4], any simple template can be arranged, via isotopy, as a *twisted braid*, which effectively means that it can be viewed as embedded in $\mathbb{R}^2 \times S^1$ so that the semiflow is in the S^1 direction (as in a forced system). Thus a simple template may be thought of as the union of a ribbon graph and a layering graph (identifying the bottom of each with the top of the other), and can therefore be specified by three pieces of

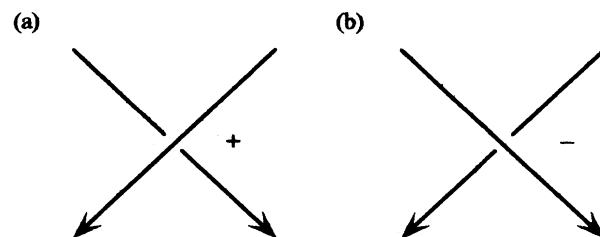


FIG. 1. Crossing convention: (a) positive, (b) negative.

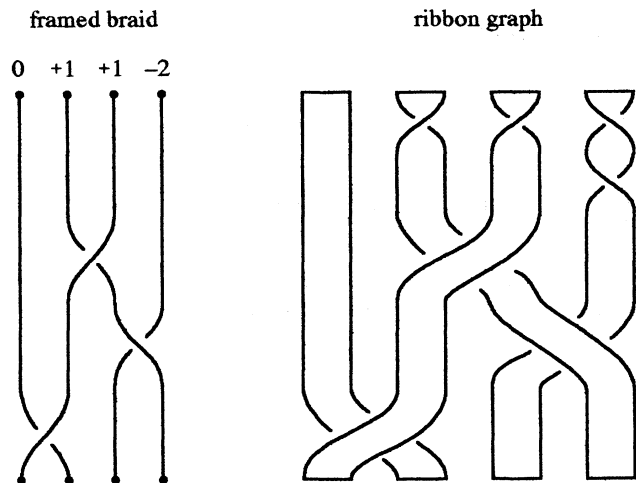


FIG. 2. Geometric representation of a framed braid as a ribbon graph.

algebraic data: a braid, a framing, and a layering array.

The decomposition of a simple template as the union of a ribbon graph and a layering graph is not unique; the line between the bottom of the ribbon graph and the top of the layering graph cannot be recovered once the graphs have been identified. In fact, any layering graph can be made to look standard near the branch set at the cost of adding a new ribbon graph at the top (this is achieved by a sequence of branch moves which are like Type-II Reidemeister moves, as in Fig. 4). Tacking this new ribbon graph onto the bottom of the old one gives a new decomposition with standard layering graph. This decomposition is unique up to a horizontal flip. Thus the template can be specified by the framed braid (unique up to a flip) associated with the ribbon graph in the standard decomposition.

The process of obtaining the standard decomposition of a simple template from a nonstandard one (typical for drawings in the literature) is illustrated in Fig. 5 for a series of two-branch templates. This also shows the associated ribbon graph (framed braid) and its “linking

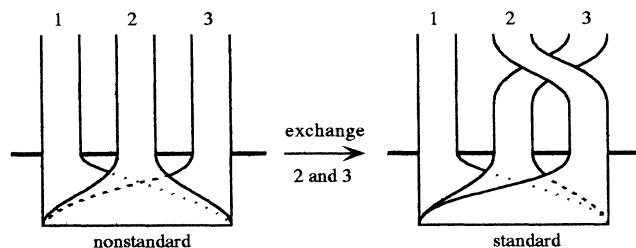


FIG. 4. Moving a layering graph to standard form: back to front.

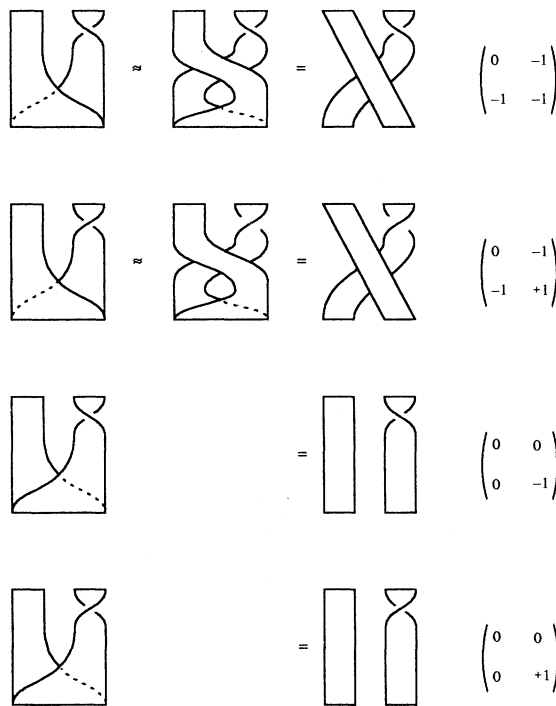


FIG. 5. Examples of two-branch templates, their standard forms, ribbon graphs, and linking matrices.

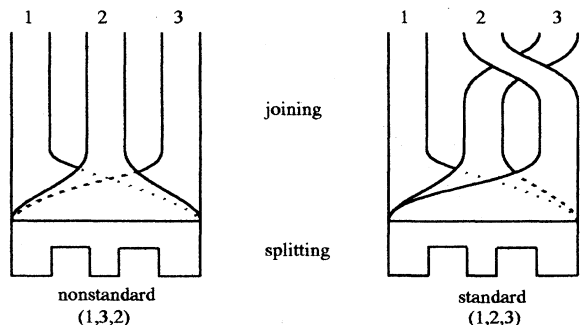


FIG. 3. Layering graph: (a) nonstandard, (b) standard.

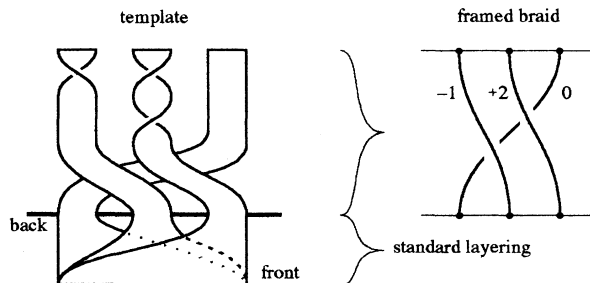


FIG. 6. A three-branch template as a framed braid.

matrix," defined below. In drawing templates, we usually confine the expanding part of the semiflow to the joining part of the layering graph, and so the branches of the layering graph get wider before they are glued at the branch line, whereas the branches of the ribbon graph are of uniform width. We also typically put the ribbon graph (with the local torsion as a series of half twists

at the top) above the joining part of the layering graph, omitting the splitting part. An example of a (standard) three-branch template is shown in Fig. 6.

The *linking matrix* of a framed braid with k strands (or its associated template) is the symmetric $k \times k$ matrix $B = (b_{ij})$ defined by

$$b_{ij} = \begin{cases} \text{the framing of the } i\text{th strand if } i = j \\ \text{the sum of the crossings between the } i\text{th and } j\text{th strands if } i \neq j. \end{cases} \quad (1)$$

Thus B describes the algebraic linking of the branches within the template (but not the subtler geometric linking), as well as the local torsion of each branch. For the example shown in Fig. 6 the linking matrix is

$$B = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 0 \end{pmatrix}.$$

From the linking matrix of a template, one can easily compute the linking numbers (or more generally the relative rotation rates [6]) of any two periodic orbits in the underlying flow. For example, the period-1 orbits corresponding to the i th and j th branches of the template have linking number $b_{ij}/2$ or $(b_{ij} + 1)/2$, according to whether b_{ij} is even or odd. (Thus the template matrix of Mindlin *et al.* [1] can be recovered from our linking matrix by adding one to any off diagonal odd entry.) For higher-order orbits, there is a simple algorithm once one has set up the appropriate symbolic dynamics (see [8] for computer codes).

The linking matrix also gives the permutation of the strands of the framed braid. At the top of the braid, the strands are ordered from 1 to k . At the bottom, each strand occupies some possibly new position. The new ordering, or *permutation* σ_B , of the strands is given by

$$\sigma_B(i) = i + [n_{\text{odd}}(b_{ij}) \text{ with } j > i] - [n_{\text{odd}}(b_{ij}) \text{ with } j < i], \quad (2)$$

where $n_{\text{odd}}(b_{ij})$ is the number of odd entries b_{ij} . Informally, to calculate the final position of the i th strand, we examine the i th row of the linking matrix, adding the number of odd entries to the right of the diagonal element to i , and subtracting the number of odd entries to

the left. For example, for the template shown in Fig. 6 we find that $\sigma_B(1) = 1 + 1 = 2$, $\sigma_B(2) = 2 + 1 - 0 = 3$, and $\sigma_B(3) = 3 - 2 = 1$, and so the permutation σ_B is the 3-cycle $(1, 2, 3)$.

Note that for a template, the layering array of Ref. [1] can be recovered from σ_B . In particular, adjusting the notation of Ref. [1] so that the branches are labeled from 1 to k rather than 0 to $k - 1$, the layering array is just $(\sigma_B^{-1}(1), \dots, \sigma_B^{-1}(k))$.

Finally, we show how to calculate the linking matrix of a template from low-order periodic orbits, which can be extracted from an experimental time series (cf. [1,5]). Let a_{ij} denote the linking number of the period-1 orbits corresponding to the i th and the j th branch, and let s_{ij} denote the (unique) self-rotation rate [6] of the period-two orbit which traverses the i th and j th branches. By convention $a_{ii} = s_{ii}$ is just the local torsion of the i th branch. Both a_{ij} and s_{ij} are computable from experimental data. Now

$$b_{ij} = \min(a_{ij}, s_{ij}), \quad (3)$$

as is readily shown by considering the 2-braid corresponding to the i th and j th branches.

In summary, we show that a "homological" classification of strange attractors is provided by a single framed-braid linking matrix. This single matrix appears to offer some theoretical and computational advantages over the original template matrix and layering array characterization [1].

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perimental Approach to Nonlinear Dynamics and Chaos (Addison-Wesley, New York, in press). See Appendix G by A. Lorentz, N. Tuffillaro, and P. Melvin for computer codes to calculate relative rotation rates.