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LETTERS TO THE EDITOR

TORUS DOUBLING AND CHAOTIC STRING VIBRATIONS: EXPERIMENTAL RESULTS

1. INTRODUCTION

Non-linear and chaotic vibrations are well documented in mechanical systems [1]; however, to the best of our knowledge (and somewhat to our surprise) chaotic oscillations have not previously been noted in experimental studies of string vibrations. Two recent theoretical studies [2, 3] predict periodic, quasi-periodic, and chaotic vibrations when strings are subject to large forcing amplitudes. In particular, Johnson and Bajaj [2] illustrated a *torus doubling* [4, 5] transition to chaos in a single-mode model (coupled Duffing oscillators) of a linear-elastic string.

In this letter we describe a simple experiment that illustrates this torus doubling transition to chaos in a string. Additional non-linear and chaotic behavior is also observed (hysteresis, chaotic transients, period doubling, etc., . . .), but we will focus on torus doubling as it is easily observable and persists for a wide range of operating parameters.

2. EXPERIMENT

Our experimental rig is similar to the one described by Gough [6]. A tungsten wire is mounted between two heavy brass anchors. An electromagnet (or large permanent magnet) is placed at the wire's midpoint, exciting vibrations in the fundamental mode when a sinusoidal current near the primary resonance passes through the wire. Both the horizontal (X) and vertical (Y) string displacements are monitored with a pair of inexpensive slotted optical sensors consisting of a LED and phototransistor in a U-shaped plastic housing [7]. The X–Y string displacement can be directly viewed and digitized, or a lock-in amplifier can be employed to average over *fast oscillations*, providing the envelope of the amplitude-modulated string vibrations. Care must also be taken to isolate the rig mechanically and acoustically. Our system is mounted on a Newport floating optical table and a cover provides acoustical isolation. Typical experimental parameters are listed in Table 1.

Both Fourier analysis and *Poincaré sections* [1] are essential as real-time diagnostics and for trajectory identification. A Tektronics 1L5 spectrum analyzer provides the former,

TABLE 1

Experimental string parameters (typical values)

Description	Value
Length	80 mm
Mass per unit length	0·59 g/m
Diameter	0·2 mm
Primary resonance	1 kHz
Range of hysteresis	300 Hz
Magnetic field strength	0·2 T
Current	0–2 A
Maximum displacement	3 mm
Damping (β) [3]	0·067

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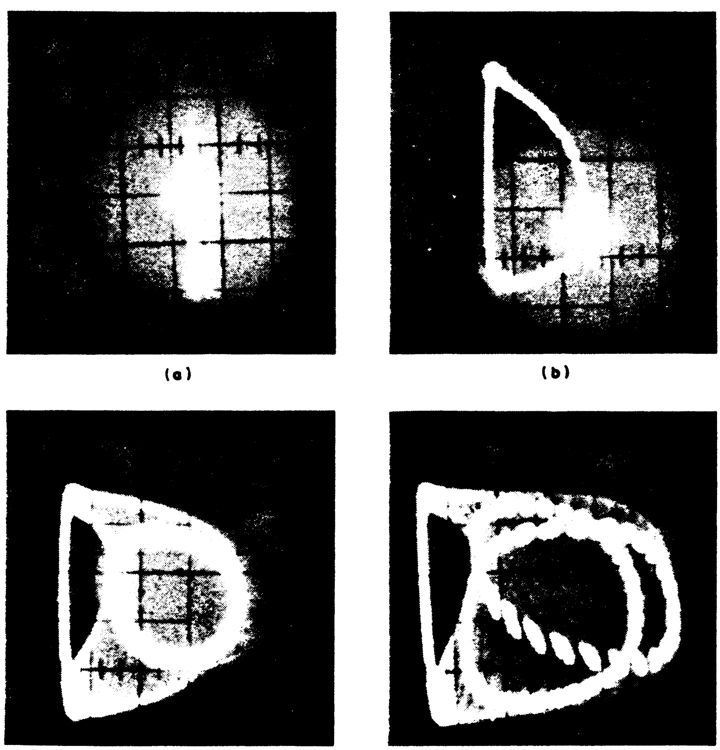
while Poincaré sections are easily obtained by sampling the horizontal and vertical string displacement once each period (stroboscopic map). A circuit converts the sinusoidal forcing function to a 50 μs pulse which is then used to trigger the beam intensity of a storage oscilloscope. A single dot appears on the screen showing the $X - Y$ string displacement sampled once each period, and the sampling phase is adjusted with a delay line. As illustrated in Figure 1, both the string displacement and the Poincaré section (bright dot) are visible at high beam intensities. Only the Poincaré section is invisible when the oscilloscope's beam intensity is decreased (Figure 2). Different attractors and qualitative changes in string motion (bifurcations) are easily identified with these tools.

3. RESULTS

We examined the orbits and bifurcation sequences for a wide range of amplitudes and detunings about the primary resonance. With either increasing forcing amplitude or detuning we typically see the bifurcation scheme illustrated by the Poincaré sections in Figures 1 and 2. Periodic planar motion ($X(t) = 0$) appears (Figure 1(a)) at small detunings from the primary linear resonance frequency (and moderate forcing amplitudes). This is followed by an elliptical (non-planar) periodic orbit (Figure 1(b)). At moderate detunings, quasi-periodic orbits occur, as indicated by the circle in the Poincaré section (Figures 1(c) and 2(a)). Next a torus doubling is observed (Figures 1(d) and 2(b)), and at slightly greater detunings the motion is chaotic (Figures 2(c) and 2(d)). To confirm that these orbits are chaotic attractors their *correlation dimension* [8] is computed from a digitized



(a) (b) (c) (d)

Figure 1. String displacement recorded by optical sensors. The bright spot in each photograph shows where the Poincaré section is recorded. (a) Planar periodic; (b) non-planar periodic; (c) quasi-periodic; (d) torus doubling of quasi-periodic motion. Only half of the displacement amplitude ($X(t) > 0$) falls within the linear range of the detectors.

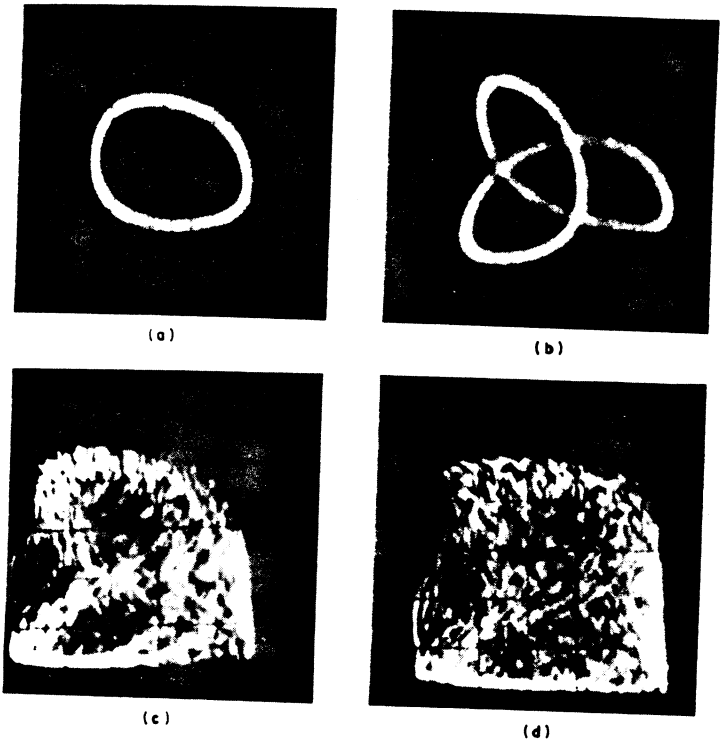


Figure 2. Poincaré sections illustrating torus doubling transition to chaos. The sequence of pictures is in order of increasing forcing frequency. (a) Quasi-periodic; (b) torus doubled quasi-periodic motion; (c) chaotic motion soon after torus doubling; (d) chaotic motion with correlation dimension 2·7.

time series. Correlation dimensions [8] between 2 and 3 are obtained for the attractors at the end of the torus doubling cascade (Figures 2(c) and 2(d)), confirming the chaotic nature of the string motion.

To summarize, we observe the following bifurcation sequence: *periodic → quasi-periodic → chaotic → quasi-periodic → periodic.* As noted above, an inverse cascade of torus doublings is often observed above the chaotic regime. We never observed more than four torus doublings in the quasi-periodic regime before the onset of chaotic motion. The exact details of a bifurcation sequence depend on the specific operating parameters. For instance, a sweep of the detuning frequency with small forcing amplitudes may result in a sequence of the form *periodic → quasi-periodic → periodic.* And, of course, at very low forcing amplitudes not even quasi-periodic motion is observed.

Torus doublings and, in addition, torus mergings, are observed for excitation frequencies near the second harmonic. The torus mergings arise from a single period doubling that occurs from the period one (non-planar) orbit. For excitations near the secondary resonance the following bifurcation scheme is commonly observed: *period one → period two → two separate tori → tori merging → tori doubling.* The single period doubling and resultant tori merging seems to arise from the interaction of the two separate spatial modes of the linear approximation.

4. CONCLUSIONS

Our experiments show good qualitative agreement with the theoretical results described by Johnson and Bajaj [2] for a torus doubling transition to chaos in strings. The apparatus is simple enough to be useful for general demonstrations of non-linear and chaotic phenomena. As emphasized earlier [3], perhaps the most important application of experiments with strings will be in testing ideas about *spatial-temporal* chaos [9]. By simply increasing the forcing frequency, the system can be varied from a single-mode to a many-modes problem. The single-mode regime is well described by a low-dimensional model; however, with the addition of new spatial modes at high forcing frequencies, significant new bifurcation schemes (and degrees of freedom) are expected to come into play as many modes become simultaneously unstable.

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