Torsional parametric oscillations in wires

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Abstract. Quite some time ago, Melde (experimentally) and Rayleigh (theoretically) investigated one of the earliest examples of a parametric oscillator. They showed that a periodic modulation of a string's length (longitudinal excitation) gives rise to transverse string vibrations. In this paper it is shown that a similar effect is observed when a periodic torsional modulation is applied to a wire; i.e., twisting a wire can parametrically excite transverse vibrations.

Riassunto. Molti anni fa Melde (sperimentalmente) e Rayleigh (teoricamente) studiarono uno dei primi esempi di oscillatore parametrico. Essi provarono che la modulazione periodica della lunghezza di una corda (eccitazione longitudinale) da luogo a vibrazioni trasversali della corda. In questo articolo si mostra che un effetto simile si osserva quando una modulazione periodica di torsione venga applicata ad un filo: la torsione di un filo puó eccitare parametricamente vibrazioni trasversali.

The essential features of a parametric oscillator are illustrated by a child on a swing. As shown in figure 1, during the down swing the child raises herself and effectively decreases the length from the pivot to the centre of mass. On the up swing, the centre of mass is again lowered. In effect, the child is periodically modulating the length of a physical pendulum, and to efficiently pump energy into the swing the child must stretch and stoop at twice the natural frequency. This illustrates a prominent feature of a parametric oscillator: to wit, a parametric resonance is strongest when the frequency of parametric excitation is twice the natural frequency [1].

Parametric oscillators are commonly found in both the playground and the laboratory. A parametric oscillation can occur when the parameter of a system is modulated; parametric devices play an essential role in several applications ranging from solid-state amplifiers to frequency converters in lasers [2]. The linear theory of parametric oscillators is generally described by differential equations of the form

$$\ddot{q} = -\omega^2(t)q \qquad \omega(t+T) = \omega(t)$$

where the frequency, ω, is a periodic function of time. For our example of the swing, the child modulates the length, and hence frequency, of a physical pendulum.

One of the first experimental studies of a parametric oscillator is Melde's experiment, in which transverse vibrations in a string are established by a purely longitudinal forcing. This forcing is achieved by connecting one end of the string to a tuning fork moving parallel to the string. In effect, the length (and hence tension) of the string is modulated periodically, and Melde discovered that this modulation leads to transverse string oscillations. This early example of a parametric oscillator was analysed theoretically by Rayleigh [3]; improvements to the theory and the experiment were achieved by Stephenson [4] and Raman [5] respectively.

Wires are often used as strings in experimental studies of string oscillations. Now wires can support both torsional modulations as well as axial modulations. This naturally raises the question: can torsional modulations in wires parametrically excite transverse oscillations? Or, more simply put, can a child swing by twisting. In this paper we will show that indeed this is the case. Basically, the torsional modulations lead to a second-order change in the string's length (and tension) so that torsional modulations can excite transverse oscillations similar to those discovered by Melde and Rayleigh. Although rather obvious, the possibility of torsional parametric excitation in wires does not seem to have been previously noted.

What is not obvious is: exactly how do torsional oscillations change the length of a wire (or rod of elastic material)? Indeed linear theory predicts that no change in length occurs, which is possibly the reason why this parametric mechanism is not previously mentioned. However, early experimental studies by Poynting [6] showed that for many wires, twisting actually makes them longer! The problem was solved

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theoretically only in 1953 by Rivlin [7] who showed, in agreement with Poynting's experimental results, that the extension in the rod is proportional to the square of the twist [8]. Rivlin's results along with some fascinating historical remarks on the deformation of rods subject to twisting are provided by Truesdell [9].

The plan of this paper is now clear, using Poynting's and Rivlin's result—namely, that extension is proportional to the square of the twist—a simple theory of torsional parametric oscillations in wires is constructed. As with many parametric oscillations we will see that these parametric oscillations are modelled by Mathieu's equation. Both the theoretical treatment and experimental observation of torsional parametric oscillations in wires is easily accessible to undergraduates and provides a lesson on exactly how non-obvious some simple problems in continuum mechanics can be.

The linear wave equation for transverse string displacements is [10]:

$$\partial^2 y/\partial t^2 = c^2(t)(\partial^2 y/\partial x^2) \qquad c^2(t) = T(t)/\mu. \tag{1}$$

Parametric effects are allowed for by the possible time dependence of $c(t)$, the transverse wave speed, which in turn depends on the tension $T(t)$ and the mass density μ. The transverse direction of the string is denoted by y and the longitudinal direction by x. The investigation of equation (1) is considerably simplified by expanding the linear wave equation in the Fourier modes,

$$y(x,t) = \sum_{n=1}^{\infty} y_n(t)\sin\left(\frac{n\pi x}{l}\right). \tag{2}$$

and examining the excitations of only the first mode y_1 resulting in,

$$\ddot{y}_1 = -\omega_0^2(t)\,y_1 \qquad \omega_0 = \frac{c(t)\pi}{l}. \tag{3}$$

The dot denotes differentiation WRT t, and l is the string length in its reference configuration.

In order to study torsional parametric excitations in wires the functional dependence of $\omega_0(t)$ in a wire subject to twisting must be specified. Following the experimental results of Poynting and the theoretical derivations of Rivlin, it will be assumed that

$$T(t) = T_0 + \gamma\theta^2 \tag{4}$$

where θ is the degree of angular displacement and γ is a constant depending on the first- and second-order elastic parameters as well as the geometry of the wire [11]. T_0 is the tension of the wire in its reference configuration.

Lastly, assume a simple periodic dependence for $\theta(t)$ of the form,

$$\theta(t) = \cos\omega t \tag{5}$$

where ω is the angular frequency of torsional forcing.

Substituting equations (4) and (5) in (3) yields

$$\ddot{y}_1 + \frac{\pi^2}{l^2\mu}[T_0 + \gamma\cos^2(\omega t)]\,y_1 = 0 \tag{6}$$

which is a form of Hill's equation [12].

In fact, the simple change of variables

$$\begin{aligned} \tau &= \omega t \\ \delta &= \pi^2(T_0 + \tfrac{1}{2}\gamma)/\mu l^2\omega^2 \\ \varepsilon &= \pi^2\gamma/4\mu l^2\omega^2 \\ \mu(\tau) &= y_1(t)/l \end{aligned} \tag{7}$$

allows equation (6) to be transformed into a special form of Hill's equation—the Mathieu equation. In terms of the new variables u and τ (prime now denotes differentiation WRT τ) equation (6) becomes

$$u'' + (\delta + 2\varepsilon\cos 2\tau)u = 0. \tag{8}$$

Mathieu's equation is well studied in parametric oscillators and related processes. Perturbative solutions to Mathieu's equation are presented in many standard texts [13]. Here, some of its properties relevant to parametric oscillations are summarised.

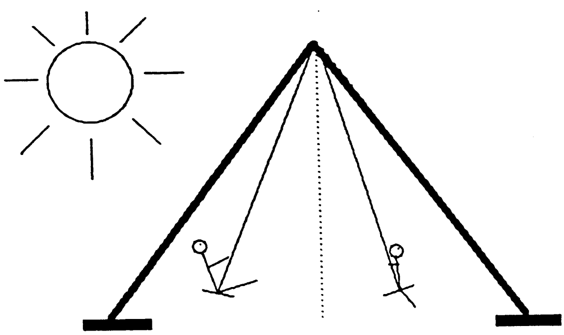
When the forcing amplitude applied to the wire is zero ($\gamma = 0$ implies $\varepsilon = 0$ in equation (7)), then equation (8) admits a simple periodic solution of the form

$$u(t) = u_0\cos(\sqrt{\delta_0}\tau + \Theta).$$

For small forcing amplitudes (ε small), equation (9) is still an approximate solution of equation (8), however the 'stability' depends critically on the parameters (δ, ε).

For fixed parameters δ and ε, equation (8) will exhibit two types of solutions, bounded (stable) orbits and unbounded (unstable) orbits. Usually, locating the unstable regions in the (δ, ε) parameter space is of interest since it is in these regions that parametric oscillations are possible. The unbounded parametric oscillations are typically asymptotic to limited cycles that arise when a non-linear term is added to equation

Figure 1. Child on a swing. The child tucks and extends her legs twice during each complete swing oscillation.



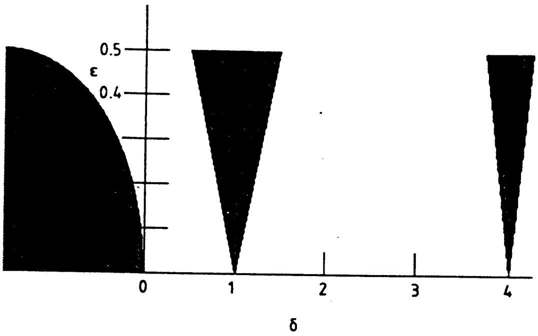


Figure 2. Schematic of stable and unstable (shaded) regions in parameter space of Mathieu's equations.

(8). This non-linear element, which must become relevant at large amplitudes, prevents solutions from becoming unbounded. However, large-amplitude considerations are not required when calculating the transition curves in the (δ, ε) plane between stable and unstable orbits. These transition curves can be calculated from perturbation theory (for small ε), numerical methods, or a combination thereof (see [13] for details). The transition curves are important since they delimit the critical parameter values necessary to sustain parametric oscillations.

Quite generally (as is shown by Floquet theory) equation (8) admits solutions of the form

$$u(\tau) = e^{\lambda\tau}\psi(\tau) \qquad (9)$$

where $\psi(\tau)$ is a periodic function of τ, and λ is the characteristic exponent determining the stability of the solution. For $\mathrm{Re}(\lambda) < 0$, the solutions are stable, and for $\mathrm{Re}(\lambda) > 0$ the solutions are unstable. The transition curves are given by the intermediate case. The transition curves showing the parameter values resulting in stable and unstable motions are illustrated in figure 2. The shaded regions show parameters resulting in unstable orbits. These shaded regions are now commonly known as 'Arnold tongues' after the mathematician V I Arnold, and play an important role in the theory of mode-locking and the transition to chaos in dissipative systems [14].

For $\varepsilon = 0$, all points on the δ axis are usually stable except for a set of integral or half-integral values of the coordinate which are the base points of the Arnold tongues [1]. These base points are subharmonics of the natural frequency, and are the initial values for which parametric oscillations are possible. For small ε, regions of parametric instability open up above the points of parametric resonance. In practice only the first few resonance points are observable since the slightest amount of friction damps out the higher-order regions of instability. Thus, in an experiment the easiest parametric resonance to observe occurs at the first subharmonic, which should be twice the natural frequency, as every child on a swing knows.

References

[1] Arnold V I 1978 *Ordinary Differential Equations* (Cambridge, MA: MIT) section 28

[2] Minorsky N 1983 *Nonlinear Oscillations* (Orlando, FL: Krieger) ch 20

[3] Lord Rayleigh 1894 *Theory of Sound* vol I (New York: MacMillan)

[4] Stephenson 1906 *Q. J. Math.* **37** 353–60

[5] Raman C V 1912 *Proc. Indian Assoc. for the Cultivation of Sci. Bull.* **6**

[6] Poynting J 1909 *Proc. R. Soc.* A **82** 546–59 (also published *Scientific Papers* (Cambridge: Cambridge University Press) article 25)

[7] Rivlin R 1963 *J. Rat. Mech. Anal.* **2** 53–81

[8] This result, according to Poynting's experimental work, is true for amplitudes in the range of four turns

[9] Truesdell C 1965 The Non-Linear Field Theories of Mechanics. *Handbuch der Physik* vol III/3 (New York: Springer)

[10] Morse P and Ingarad K 1968 *Theoretical Acoustics* (NY: McGraw-Hill) ch 4

[11] Explicit expressions for γ are given by Rivlin and Truesdell, however it is probably easier to determine γ experimentally with the methods described by Poynting

[12] Stoker J 1950 *Nonlinear Vibrations* (NY: Interscience) ch VI

[13] A detailed and elementary study of the Mathieu equation is provided Nayfeh A 1981 *Introduction to Perturbation Techniques* (New York: Wiley-Interscience) ch 11

[14] Bak P, Bohr T and Jensen M 1985 *Phys. Scr.* T **9** 50–8