

Hamiltonian Dynamics of the Swinging Atwood's Machine

Using GNU `plotutils`

Introduction

The Swinging Atwood's Machine (SAM) is a classic problem in non-linear dynamics. By transitioning from the Lagrangian formulation to the Hamiltonian formulation, we describe the system using canonical coordinates: the generalized positions (r, θ) and their corresponding conjugate momenta (p_r, p_θ) .

The Lagrangian is defined as $L = T - V$:

$$L = \frac{1}{2}(M + m)\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 - gr(M - m \cos \theta)$$

The conjugate momenta are obtained by taking the derivative of the Lagrangian with respect to the generalized velocities:

$$p_r = \frac{\partial L}{\partial \dot{r}} = (M + m)\dot{r} \quad (1)$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \quad (2)$$

Applying a Legendre transformation yields the Hamiltonian $H(r, \theta, p_r, p_\theta)$, representing the total energy of the system:

$$H = \frac{p_r^2}{2(M + m)} + \frac{p_\theta^2}{2mr^2} + gr(M - m \cos \theta)$$

Hamilton's equations of motion ($\dot{q} = \frac{\partial H}{\partial p}$ and $\dot{p} = -\frac{\partial H}{\partial q}$) then give us a system of four first-order ordinary differential equations:

$$\dot{r} = \frac{p_r}{M + m} \quad (3)$$

$$\dot{\theta} = \frac{p_\theta}{mr^2} \quad (4)$$

$$\dot{p}_r = \frac{p_\theta^2}{mr^3} - g(M - m \cos \theta) \quad (5)$$

$$\dot{p}_\theta = -grm \sin \theta \quad (6)$$

Installation of GNU `plotutils`

To run the provided Bash script, you must have the GNU `plotutils` package installed, as it provides both the `ode` mathematical integrator and the `graph` rendering utility.

Linux

Most major Linux distributions include `plotutils` in their default package repositories. Open your terminal and run the command corresponding to your distribution:

- **Ubuntu / Debian:**
`sudo apt install plotutils`
- **Fedora / RHEL:**
`sudo dnf install plotutils`
- **Arch Linux:**
`sudo pacman -S plotutils`

macOS

On macOS, the easiest and most reliable way to install `plotutils` is via the Homebrew package manager. From your terminal, run:

- **Homebrew:**
`brew install plotutils`

Windows

Because this script heavily relies on the Bash shell and UNIX-style piping, the most modern and reliable way to run it on Windows is through the **Windows Subsystem for Linux (WSL)** or a UNIX-like compatibility layer.

- **WSL (Ubuntu/Debian environment):**
Open your WSL terminal and install it using the Debian package manager:
`sudo apt install plotutils`
- **MSYS2:**
If you use MSYS2, open your terminal and install via pacman:
`pacman -S plotutils`
- **Cygwin:**
Run the Cygwin Setup installer (`setup-x86_64.exe`), search for `plotutils` in the package list, and mark it for installation.

Bash Script

The bash script below uses the `ode` and `graph` utilities. `ode` handles the numerical integration of the coupled differential equations using a Runge-Kutta-Fehlberg (RKF45) step-size control algorithm, converts the variables to Cartesian coordinates (x, y) , and pipes the output directly to `graph` to render the trajectory.

```
1 #!/bin/bash
2 # Swinging Atwood's Machine (Hamiltonian) using GNU plotutils
3
4 # We use a Here-Doc to feed the Hamiltonian model into 'ode',
```

```

5 # and pipe the x, y output directly into 'graph' to generate an
   image.
6
7 ode << 'EOF' | graph -T png -C -W 0.003 -X "x position" -Y "y
   position" -L "Swinging Atwood's Machine (Hamiltonian, M=1.05,
   m=1.0)" > sam_hamiltonian.png
8
9 # 1. System Parameters
10 m = 1.0
11 M = 1.05
12 g = 9.81
13
14 # 2. Hamilton's Equations of Motion
15 # r' = r dot
16 # th' = theta dot
17 # pr' = p_r dot (radial conjugate momentum)
18 # pth' = p_theta dot (angular conjugate momentum)
19
20 r' = pr / (M + m)
21 th' = pth / (m * r^2)
22 pr' = (pth^2) / (m * r^3) - g * (M - m * cos(th))
23 pth' = -g * r * m * sin(th)
24
25 # 3. Coordinate Transformation (Polar to Cartesian for plotting)
26 x = r * sin(th)
27 y = -r * cos(th)
28
29 # 4. Output instruction for 'ode'
30 print x, y
31
32 # 5. Initial Conditions
33 r = 1.0
34 th = 1.5707963 # Start at pi/2 (horizontal)
35 pr = 0.0 # Initial radial momentum
36 pth = 0.0 # Initial angular momentum
37
38 # 6. Integration command: step from t=0 to t=50 seconds
39 step 0, 50
40 EOF
41
42 echo "Simulation complete. Output successfully saved to
   sam_hamiltonian.png."

```

Listing 1: Bash script for Hamiltonian SAM using ode and graph

References

- Casasayas, J., Nunes, A., & Tuffillaro, N. (1990). Swinging Atwood's Machine : integrability and dynamics. *Journal de Physique*, 51(16), 1693-1702.
- Fehlberg, E. (1969). *Low-order classical Runge-Kutta formulas with stepsize control and their application to some heat transfer problems*. NASA Technical Report R-315. (The foundational text detailing the RKF45 adaptive numerical integration algorithm utilized by GNU `ode`).
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- Pujol, O., Pérez, J.P., Ramis, J.P., Simó, C., Simon, S., & Weil, J.A. (2010). Swinging Atwood Machine: Experimental and numerical results, and a theoretical study. *Physica D: Nonlinear Phenomena*, 239(12), 1067-1081.
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