Teardrop and heart orbits of a swinging Atwood's machine

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An exact solution is presented for a swinging Atwood's machine. This teardrop–heart orbit is constructed using Hamilton–Jacobi theory. The example nicely illustrates the utility of the Hamilton–Jacobi method for finding solutions to nonlinear mechanical systems when more elementary techniques fail.

A swinging Atwood's machine (SAM) is an ordinary Atwood's machine in which, however, one of the weights can swing in a vertical plane.[1] In the polar coordinates (r,θ), the total energy of SAM is

$$E=T+V$$
$$=\frac{1}{2}(m+M)\dot{r}^2+\frac{1}{2}mr^2\dot{\theta}^2+gr(M-m\cos\theta), \qquad (1)$$

where M is the mass of the nonswinging bob, m is the mass of the swinging bob, g is the gravitational acceleration, and the polar angle θ is measured from the negative vertical y axis as shown in Fig. 1(a). When the mass ratio $\mu=M/m=3$, numerical studies of SAM reveal an unexpected feature. To wit, all orbits which start at the origin (with initial conditions $r=\epsilon$, $\dot{r}=v$, $\theta=\theta_0$) *execute one symmetrical loop and return to the origin, no matter what the launch angle* (θ_0) *or speed* (v).[2] When the launch angle is small, the orbit is called a *teardrop*, and when the launch angle is larger (say $\theta_0>\pi/2$) the orbit is called a *heart* (see Fig. 1).

Since the SAM Hamiltonian is nonlinear, we generally expect that it will be impossible to find (simple) exact solutions for SAM.[3] In this paper I show that the "teardrop–heart" orbit of SAM is an exception to this rule. Specifically, I use Hamilton–Jacobi theory to construct an exact solution for the teardrop–heart orbit. This example would make a nice addition to an advanced mechanics class since it illustrates the power of the Hamilton–Jacobi method to arrive at an exact solution which apparently eludes more elementary methods.

Recall that the game of Hamilton–Jacobi theory is to find some canonical transformation which will separate variables in the Hamilton–Jacobi equation. A fortuitous choice for the SAM Hamiltonian is the point transformation

$$r=\frac{1}{2}(\xi^2\eta^2), \qquad (2)$$

$$\theta=2\arctan[(\xi^2-\eta^2)/2\xi\eta]. \qquad (3)$$

This transformation compresses the angular variable by a factor of 2, and then changes from polar coordinates to parabolic coordinates with parabolas centered about the horizontal x axis.[4] The inverse transformation is

$$\xi=\sqrt{r}\sqrt{1+\sin(\theta/2)}, \qquad (4)$$

$$\eta=\pm\sqrt{r}\sqrt{1-\sin(\theta/2)}. \qquad (5)$$

If I further set $M=3m$ (and $m=1$) as is done throughout the rest of this paper I find that the generalized momenta in the new coordinate system are

$$p_\xi=4\dot{\xi}(\xi^2+\eta^2)=\frac{\partial S}{\partial\xi},\quad p_\eta=4\dot{\eta}(\xi^2+\eta^2)=\frac{\partial S}{\partial\eta} \qquad (6)$$

from which the (time-dependent) Hamilton–Jacobi equation is calculated as

$$\frac{\partial S}{\partial t}+\frac{1}{(\xi^2+\eta^2)}\left\{\frac{1}{8}\left[\left(\frac{\partial S}{\partial\xi}\right)^2+\left(\frac{\partial S}{\partial\eta}\right)^2\right]+2g(\xi^4+\eta^4)\right\}=0. \qquad (7)$$

The equation for the generating function is now separable and can be expressed in the form

$$S(\xi,\eta,t)=S_\xi(\xi)+S_\eta(\eta)+S_t(t). \qquad (8)$$

Separating the time and space parts of Eq. (7) find that the first separation constant is just the total energy, E:

$$\frac{\partial S}{\partial t}=-E, \qquad (9)$$

$$\frac{1}{(\xi^2+\eta^2)}\left\{\frac{1}{8}\left[\left(\frac{\partial S}{\partial\xi}\right)^2+\left(\frac{\partial S}{\partial\eta}\right)^2\right]+2g(\xi^4+\eta^4)\right\}=E. \qquad (10)$$

Integration of Eq. (9) yields

$$S_t(t)=-Et. \qquad (11)$$

Equation (10) can be written as

$$\frac{1}{8}\left[\left(\frac{\partial S}{\partial\xi}\right)^2+\left(\frac{\partial S}{\partial\eta}\right)^2\right]+2g(\xi^4+\eta^4)=E(\xi^2+\eta^2). \qquad (12)$$

Next I can separate the spatial part of the Hamilton–Jacobi [Eq. (12)] equation giving

$$\left(\frac{\partial S}{\partial\xi}\right)^2+16g\xi^4-8E\xi^2=I, \qquad (13)$$

$$-\left(\frac{\partial S}{\partial\eta}\right)^2-16g\eta^4+8E\eta^2=I, \qquad (14)$$

where I is the spatial separation constant. The complete solution to the Hamilton–Jacobi equation is then given by

$$S=\int(I+8E\xi^2-16g\xi^4)^{1/2}d\xi$$
$$+\int(-I+8E\eta^2-16g\eta^4)^{1/2}d\eta-Et. \qquad (15)$$

To "solve" the original problem I do not need to integrate Eq. (15) directly, but rather the *orbit equation*

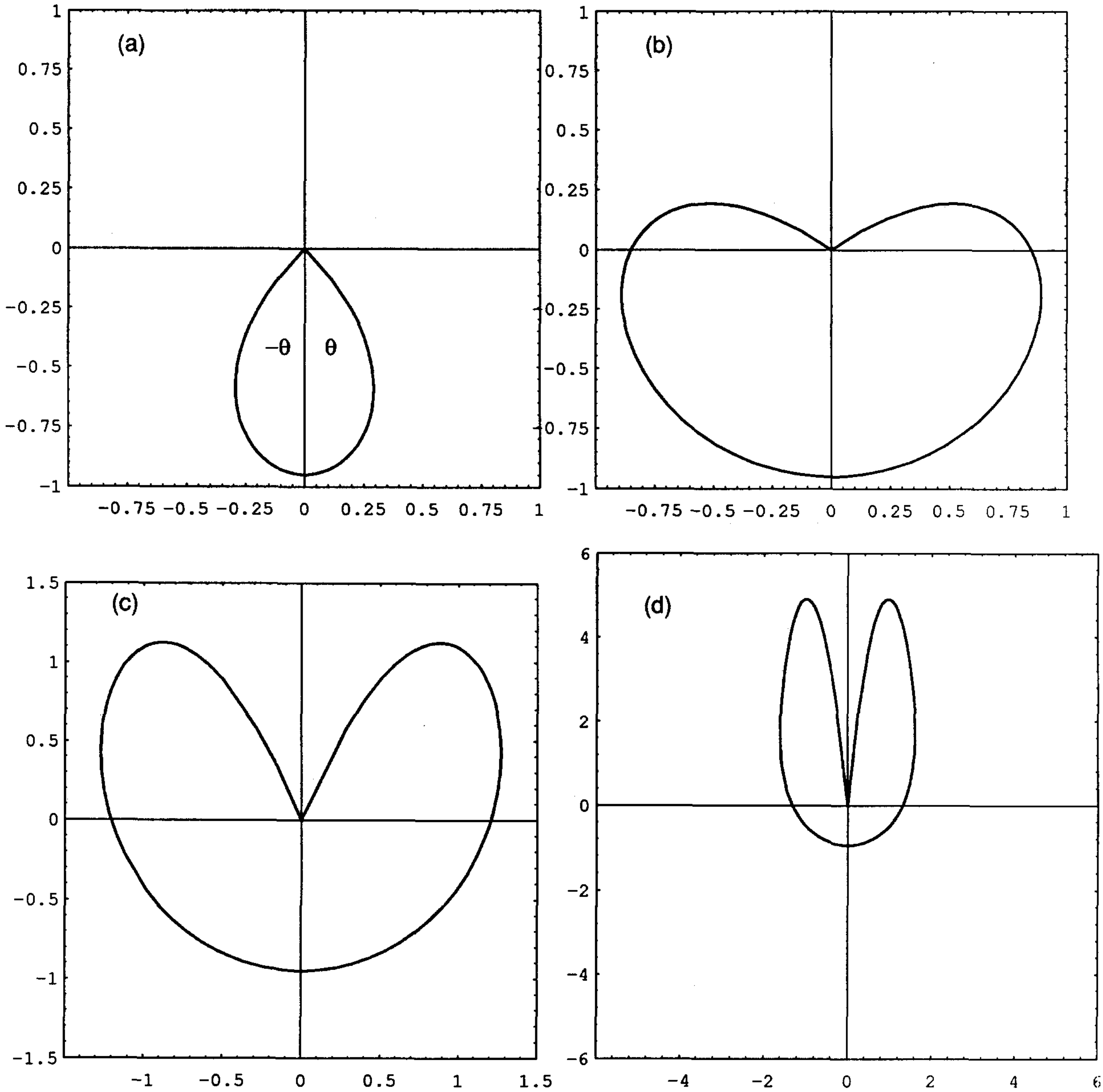

Fig. 1. (a) Teardrop, (b), (c) heart, and (d) a rabbit ear orbit of a SAM when $\mu = M/m = 3$.

$$\frac{\partial S}{\partial I} = \frac{1}{2}\int \frac{1}{(I + 8E\xi^2 - 16g\xi^4)^{1/2}}\,d\xi - \frac{1}{2}\int \frac{1}{(-I + 8E\eta^2 - 16g\eta^4)^{1/2}}\,d\eta = \alpha \qquad (16)$$

which provides an implicit relation between ξ and η (and, therefore, r and θ), and the parametrized *trajectory equation*

$$\frac{\partial S}{\partial E} = \int \frac{4\xi^2}{(I + 8E\xi^2 - 16g\xi^4)^{1/2}}\,d\xi + \int \frac{4\eta^2}{(-I + 8E\eta^2 - 16g\eta^4)^{1/2}}\,d\eta = t + \beta, \qquad (17)$$

where α and β are constants of integration which may, in general, be a complicated mix of the original initial conditions.

Just as important, I can also solve for the new constant of the motion, I, by adding Eqs. (13) and (14), and then eliminating the energy constant to get

$$\frac{1}{8} I = 2(\xi^2 + \eta^2)(\eta^2\xi^2 - \xi^2\dot{\eta}^2) + 2g\xi^2\eta^2[(\xi^2 - \eta^2)/(\xi^2 + \eta^2)]. \qquad (18)$$

In the original polar coordinate system [apply Eqs. (4) and (5) to Eq. (18)] the new constant of motion reads

$$\frac{1}{4} I(r,\dot{r},\theta,\dot{\theta}) = r^2\dot{\theta}\left[\dot{r}\cos\left(\frac{\theta}{2}\right) - \frac{r\theta}{2}\sin\left(\frac{\theta}{2}\right)\right] + gr^2\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right). \qquad (19)$$

In principle the orbit and trajectory equations arising from the generating function represent a complete solution to the original problem—in this case finding the motion of SAM when $\mu=3$. In practice, though, there are still two difficult obstacles to overcome. First, actually finding explicit integrals to the orbit and trajectory equations may not be possible, and second, the resulting relations are often, at best, only implicit function which cannot be inverted for the relevant variables.

In this particular example, the first of these obstacles can be overcome—the orbit and trajectory equations can be explicitly integrated with elliptic functions. However, instead of using special functions to arrive at a solution, I would next like to turn my attention to a specific solution—the teardrop–heart orbits—and show that for this special class of orbits it is possible to obtain a simple and elementary solution for the motion of SAM.

The key observation used in arriving at this elementary solution is as follows: *for all teardrop–heart orbits the constant of motion "I" is equal to zero.* To see why this last observation is true consider the teardrop–heart orbit as it crosses the negative vertical axis (see Fig. 1). At the axis, $\theta=0$ and $\dot{r}=0$. Plugging these values into Eq. (19) it follows that $I(r,\dot{r},\theta,\dot{\theta})=0$. Next I show that when $I=0$ the integrals in the orbit and trajectory equations can be solved by elementary functions, and the resulting functions can be inverted to arrive at a simple orbit equation relating r and θ.

Examining the special case $I=0$, the orbit equation [Eq. (16)] becomes

$$2\alpha=\int\frac{1}{(8E\xi^2-16g\xi^4)^{1/2}}\,d\xi-\int\frac{1}{(8E\eta^2-16g\eta^4)^{1/2}}\,d\eta \tag{20}$$

which can be integrated straight away to give[5]

$$2\alpha=\frac{1}{\sqrt{8E}}\ln\left(\frac{1-\sqrt{1-(2g/E)\xi^2}}{1-\sqrt{1-(2g/E)\eta^2}}\,\frac{\eta}{\xi}\right). \tag{21}$$

After some additional algebraic manipulations I find the implicit orbit equation in ξ and η to be

$$a=\left(\frac{1-\sqrt{1-k\xi^2}}{1-\sqrt{1-k\eta^2}}\right)\frac{\eta}{\xi}, \tag{22}$$

where the positive constants a and k are defined by

$$a=e^{2\alpha\sqrt{8E}},\quad k=\frac{2g}{E}. \tag{23}$$

Using the inverse point transformation Eqs. (4) and (5) applied to Eq. (22) I get the implicit orbit equation in polar coordinates as

$$\pm a=\frac{\sqrt{1-\sin(\theta/2)}}{\sqrt{1+\sin(\theta/2)}}\,\frac{\sqrt{1-kr(1+\sin\theta/2)}-1}{\sqrt{1-kr(1-\sin\theta/2)}-1}. \tag{24}$$

At this point it is helpful to pause and check this result by considering the limiting case of a teardrop orbit, namely, the solution of Eq. (24) in the limit of small θ. Expanding Eq. (24) in θ about zero I find[6]

$$\pm a=1+\frac{\theta}{2\sqrt{1-kr}}+O(\theta^2). \tag{25}$$

Or, written another way

$$r(\theta)\approx r_0\left[1-\left(\frac{\theta}{\theta_0}\right)^2\right],\quad r_0=1/k,\quad \theta_0=2(-1\pm a) \tag{26}$$

which indeed produces the teardrop-shaped orbit shown in Fig. 1(a).

Now I return to the main stream of my presentation by attempting to find an explicit orbit equation. Rewrite Eq. (22) in the form $a\xi\sqrt{1-k\eta^2}-\eta\sqrt{1-k\xi^2}=a\xi-\eta$, square both sides, and collect like terms to discover $2a-k\xi\eta-ka^2\xi\eta=2a\sqrt{1-k\xi^2-k\eta^2+k^2\xi^2\eta^2}$. Next notice [from Eqs. (4) and (5)] that $\xi\eta=r\cos(\theta/2)$. Evidently, $(k^2+k^2a^4-2a^2k^2)r^2\cos^2(\theta/2)+(-4ak-4a^3k)r\cos(\theta/2)+(4a^2k)r[1+\sin(\theta/2)]+(4a^2k)r[1-\sin(\theta/2)=0$. Additional algebraic simplification results in the exact explicit *teardrop–heart orbit equation*

$$r(\theta)=\frac{4a}{k}\,\frac{(a^2+1)\cos(\theta/2)-2a}{(a^2-1)^2\cos^2(\theta/2)},\quad -\theta_0<\theta<\theta_0, \tag{27}$$

that is, the teardrop–heart orbit occurs for positive values of r, an this condition places the above restriction on θ where (at $r=0$)

$$\theta_0=2\arccos\left(\frac{2a}{a^2+1}\right). \tag{28}$$

Also, when the orbit crosses the negative vertical axis its length is (at $\theta=0$)

$$r_0=\frac{4a}{k}\,\frac{1}{(a+1)^2}. \tag{29}$$

That is it.

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[1]D. J. Griffiths and T. A. Abbott, "Comment on 'A surprising mechanics demonstration,'" Am. J. Phys. **60**, 951–953 (1992).

[2]N. B. Tufillaro, B. A. thesis, Reed College, Portland, OR, 1982.

[3]N. B. Tufillaro, T. Abbott, and J. Reilly, *An Experimental Approach to Nonlinear Dynamics and Chaos* (Addison–Wesley, Reading, MA, 1992).

[4]The identity of $\cos(2\theta)=(1-\tan^2\theta)/(1+\tan^2\theta)$ is useful in making this change of coordinates.

[5]The integrals are quickly done with a symbolic program such as *Mathematica* or *Axiom*. The latter program gives $\int(1/[(8E\xi^2-16g\xi_4)^{1/2}]=1/(\sqrt{8E})[\log(\sqrt{2E-4g\xi^2})-\sqrt{2E})-\log(\xi)]$ and $\int(8\xi^2)/[(8E\xi^2-16g\xi^4)^{1/2}]=(-2^{1/2}/g)(\sqrt{E-2g\xi^2})$.

[6]This expansion is easily done with a symbolic mathematics program. In *Mathematica* the command "Simplify[Series[(Eq.(24)),{θ,0,2}]]" results in $1+\theta/(2\sqrt{1-kr})+[(-2+2kr-2\sqrt{1-kr}+kr\sqrt{1-kr})\theta^2]/[8\sqrt{1-kr}(-1+kr)(1+\sqrt{1-kr})^2]+O(\theta^3)$.