Eur. J. Phys. 10 (1989) 173–177. Printed in the UK

Infinity manifold of a swinging Atwood's machine

J Casasayas†, N Tufillaro‡ § and A Nunes|| ¶

† Departament de Matemàtica Aplicada i Anàlisi, Universitat de Barcelona, Barcelona, Spain
‡ Department of Physics, Bryn Mawr College, Bryn Mawr, PA 19010, USA
|| Department de Matemàtiques, Universitat Autònoma de Barcelon, Barcelona, Spain

Received 24 March 1988

Abstract. The unbounded motions of a swinging Atwood's machine are analysed by *blowing up* the singularity at infinity. The asymptotic motion is reduced to a gradient flow on an ellipsoid. By studying the flow on this ellipsoid it is shown that the unbounded orbits oscillate either an infinite number of times or not at all, depending only on a critical value of the mass ratio.

Résumé. On analyse les mouvements non bornés d'une machine d'Atwood oscillante en faisant *exploser* la singularité á l'inifini. Le mouvement asymptotique se réduit a un flot tangent á un gradient sur un ellipsoïde. En étudiant le flot sur cet ellipsoïde, on montre que les orbites non bornées oscillent une infinité de fois ou pas du tout d'après la valeur du rapport des masses comparées á une valeur critique.

1. Introduction

Advances in the research of non-linear dynamical systems during the past two decades [1] have renewed interest in the study of low-dimensional mechanical models since they exhibit many complicated types of motion from periodic and quasi-periodic to chaotic [2, 3]. In fact, the problems posed by these simple systems can be quite hard, and as Arnold has remarked [4], 'analyzing a general potential system with two degreees of freedom is beyond the capability of modern science'. Both the analytic and numerical study of simple mechanical systems can be further complicated by the presence of *singularities*, i.e. points where the relevant differential equations are undefined. The simplest example of a singularity is the collision of two or more point particles in the Newtonian n-body problem. *Regularisation* is the name given to a host of mathematical techniques that allow one to solve for the orbits in the vicinity of a singularity [5]. Regularisation techniques are of practical interest, for instance in solving for the near collision orbits between artificial bodies and celestial bodies.

A particularly simple and geometrically appealing regularisation procedure was introduced sometime ago by McGehee, who found that by a simple change of variables it is possible to *blow up* the singular set where the differential equations are undefined [6]. Moreover, the motion on this new set, the so-called *collision manifold*, turns out to be very simple since it is usually a gradient-like flow.

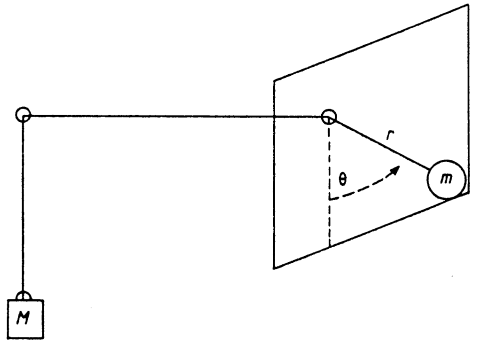

Figure 1. Swinging Atwood's machine (SAM). The configuration space variables are the polar coordinates (r, θ) of the swinging mass.

§ Current address: Department of Physics, University of Otago, Dunedin, New Zealand.
¶ On leave from Deparamento de Fisica, Universidade de Lisboa, Lisboa, Spain, partially supported by Fundacao Calouste Gulbenkian.

0143-0807/89/030173 + 05 $02.50

The motion on the collision manifold yields information about orbits that pass close to the singularity because of the continuous dependence on initial conditions of an analytic flow. A good introduction to blowing up singularities and gradient-like flows is given by Devaney [8, 9]. Some time later Lacomba and Simó pointed out that the same method could be used to analyse the unbounded motions arising from problems in celestial mechanics [10]. In essence, they showed that the unbounded orbits can also approach a singular set, which when blown up allows one to analyse the unbounded orbits by a simple geometric method.

A simple non-linear Hamiltonian system that has received considerable attention of late is the swinging Atwood's machine (SAM) [11, 12, 13]. As shown in figure 1, SAM is an ordinary Atwood's machine in which one of the masses is allowed to swing in a plane. SAM has two degrees of freedom that are taken to be the polar coordinates (r, θ) of the swinging mass. The system is of interest because it is very simple yet exhibits the full spectrum of motions from integrable to chaotic [14, 15, 16]. The equations of motion for SAM are singular when the radial coordinate r approaches 0. These *collision orbits* have been previously analysed by using the McGehee regularisation procedure [17]. In this paper we observe that the orbits are also singular when r approaches infinity. The asymptotic behaviour of these orbits turns out to be simple to understand by blowing up the singularity at infinity. Specifically, we show that if the mass ratio defined by $\mu = M/m$ satisfies $\mu \in (1/17, 1]$ then the orbits oscillate an infinite number of times about the downward $(\theta = 0)$ axis as r approaches infinity. The technique illustrated here should be useful in the study of a wide variety of Hamiltonian systems exhibiting unbounded orbits.

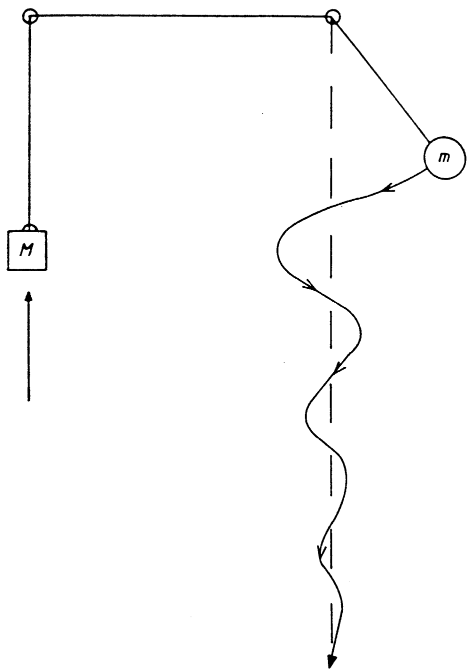

Figure 2. Typical unbounded motion of SAM undergoing damped oscillations about the downward $(\theta = 0)$ axis.

2. Unbounded orbits

The Hamiltonian for SAM is simply the sum of the kinetic and potential energy:

$$H(r, \theta, p_r, p_\theta) = T(p, q) + V(q) = \frac{p_r^2}{2(m+M)} + \frac{p_\theta^2}{2mr^2} + gr(M - m\cos\theta) \qquad (1)$$

where the canonical momenta are $p_r = (m+M)\dot{r}$, $p_\theta = mr^2\dot{\theta}$, and g is the gravitational acceleration constant. Let h denote a particular value of the energy. The kinetic energy is positive definite, making it easy to show that all motion is bounded by $r \leqslant h/(\mu - \cos\theta)g$ whenever $\mu > 1$. Unbounded motions exist only when $\mu \leqslant 1$. The region in configuration space where the unbounded orbits exist is also limited by the Hamiltonian. For instance, if $\mu = 1$ then the motion is bounded in all directions except $\theta = 0$. If $\mu < 1$ then unbounded orbits can exist when $\theta \in [-\cos^{-1}(\mu - h/gr), \cos^{-1}(\mu - h/gr)]$.

Extensive numerical simulations indicate that all unbounded orbits behave pretty much as illustrated in figure 2. As r approaches infinity the orbits undergo damped oscillations about the downward $(\theta = 0)$ axis. In other words, in the limit as $t \to \infty$ and $r(t) \to \infty$, the angle $\theta(t) \to 0$. It is also observed that for very small mass ratios the oscillations cease. In either case $\theta(t)$ still approaches zero when $0 < \mu \leqslant 1$. In §3 we will explain these observations by blowing up the singularity at infinity.

3. Infinity manifold

Hamilton's equations obtained from (1) are

$$\begin{aligned}
\dot{r} &= \frac{\partial H}{\partial p_r} = \frac{p_r}{(m+M)} \\
\dot{\theta} &= \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mr^2} \\
\dot{p}_r &= -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - g(M - m\cos\theta) \\
\dot{p}_\theta &= -\frac{\partial H}{\partial \theta} = -mgr\sin\theta.
\end{aligned} \qquad (2)$$

In order to solve for the unbounded asymptotic motions of SAM we consider a generalisation of the change of variables introduced by Lacomba and Simó [10]. In effect, our change of variable blows up infinity by taking $\rho = 1/r$ and slows down the time by adjusting to a new time scale $\tau \propto t/\sqrt{r}$. These two steps, magnifying around the singularity and slowing down time, are the key ingredients of any McGehee-type regularisation procedure.

Specifically, consider the *non-canonical* change of variables $(r, \theta, p_r, p_\theta, t) \to (\rho, \theta, v, u, \tau)$ defined by

$$\rho = r^{-1} \quad \theta = \theta \quad v = p_r r^{-1/2} \quad u = p_\theta r^{-3/2}$$

$$dt/d\tau = \rho^{-1/2}.$$

The SAM equations (2) become

$$\rho' = \frac{-\rho v}{(M+m)} \qquad \theta' = \frac{u}{m}$$

$$v' = \frac{u^2}{m} - \frac{v^2}{2(M+m)} - g(M - m\cos\theta) \tag{3}$$

$$u' = \frac{-3uv}{2(M+m)} - mg\sin\theta$$

where the primes indicate differentiation with respect to τ. The energy relation becomes

$$\rho h = \frac{v^2}{2(M+m)} + \frac{u^2}{2m} + g(M - m\cos\theta). \tag{4}$$

Now, the vector field (3) is analytic on the invariant boundary $\rho = 0$, denoted by N_h and called the *infinity manifold*. The energy relation (4) shows that N_h is a generalised ellipsoid given by

$$N_h = \{(\rho, \theta, v, u): \rho = 0,\ v^2/2(m+M) + u^2/2m + g(M - m\cos\theta) = 0\} \tag{5}$$

and is shown in figure 3. Note that from (5) we have

$$u^2/2m + v^2/2(m+M) = gm(\cos\theta - \mu)$$

where $\mu = M/m$. So, $\cos\theta \geqslant \mu$ and, in particular, $N_h = \varnothing$ *for* $\mu > 1$, $N_h = (\rho = 0, \theta = 0, u = 0, v = 0)$ for $\mu = 1$ and N_h is the ellipsoid represented in figure 3 for $\mu < 1$. Thus the ellipsoid only exists for $\mu < 1$.

The flow on N_h is given by (7) when we take the asymptotic limit of (3) by setting $\rho = 0$:

$$\theta' = u/m \qquad v' = 3u^2/2m$$
$$u' = -3uv/2(m+M) - mg\sin\theta. \tag{6}$$

From (6), making the change $d\tau/d\bar{\tau} = m$ and denoting by an overbar the derivative with respect to $\bar{\tau}$, we have

$$\bar{\theta} = u \qquad \bar{v} = \tfrac{3}{2}u^2$$
$$\bar{u} = -3uv/2(1+\mu) - m^2 g\sin\theta \tag{7}$$

and the energy relation on the infinity manifold is

$$v^2/2(m+M) + u^2/2m + g(M - m\cos\theta) = 0. \tag{8}$$

To visualise the flow on the infinity manifold we must find the equilibrium points of (7) and analyse the local motions near the equilibrium points by

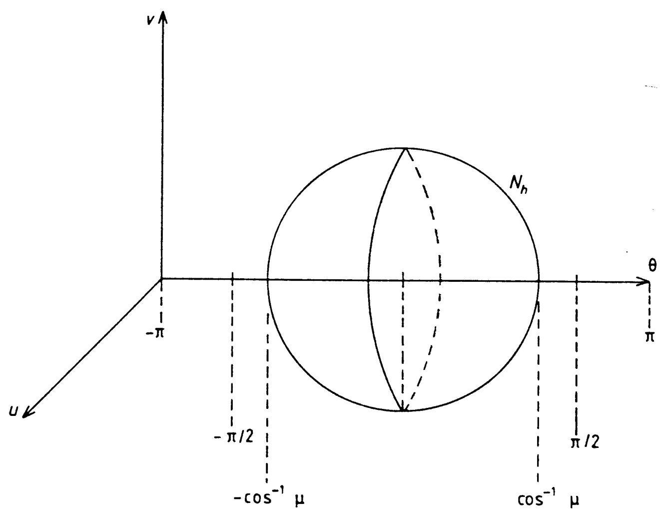

Figure 3. The infinity manifold N_h of SAM. Note that if $\mu \to 1$ then $\cos^{-1}\mu \to 0$, and when $\mu \to 0$, $\cos^{-1}\mu \to \pi/2$.

Table 1. Here we use the notation $v^{\pm} = \pm[2(m^2 - M^2)g]^{1/2}$.

Equilibrium point	Characteristic exponents on N_h	off N_h	Dimensions of W^u, W^s	on N_h	on $H=h$	Type on N_h
$P^+(0)$	$\frac{1}{4}\left[\frac{-3v^+}{1+\mu} \pm \left(\frac{9v^2}{(1+\mu)^2} - 16m^2g\right)^{1/2}\right]$	$\frac{-v^+}{m+M}$	W^u	0	0	sink
			W^s	2	3	
$P^-(0)$	$\frac{1}{4}\left[\frac{-3v^-}{1+\mu} \pm \left(\frac{9v^2}{(1+\mu)^2} - 16m^2g\right)^{1/2}\right]$	$\frac{-v^-}{m+M}$	W^u	2	3	source
			W^s	0	0	

calculating the associated eigenvalues. By setting $\bar{\theta} = \bar{v} = \bar{u} = 0$, the equilibrium points are found to be

$$P^{\pm}(0) = (\rho = 0, \theta = 0, v = \pm[2(\theta^2 = M^2)g]^{1/2}, u = 0) \tag{9}$$

and the eigenvalues are

$$\lambda_{\pm} = \frac{1}{4}\left[\frac{-3v^{\pm}}{1+\mu} \pm \left(\frac{9v^2}{(1+\mu)^2} - 16m^2g\right)^{1/2}\right] \tag{10}$$

where we have used the notation

$$v^{\pm} = \pm[2(m^2 - M^2)g]^{1/2}. \tag{11}$$

The $P^+(0)$ equilibrium point is a sink on N_h while the $P^-(0)$ is a source. Since the flow is gradient like with respect to v, all the orbits not on an equilibrium point must approach $P^+(0)$. Unbounded orbits of SAM approach the infinity manifold via the stable manifold, W^s, and orbits arive from infinity via the unstable manifold, W^u, emanating from N_h. The eigenvalues and relative dimensions of the stable and unstable invariant manifolds are summarised in table 1.

Furthermore, from equation (10) we see that if $0 < \mu \leqslant 1/17$ then the sinks and sources on N_h have characteristic exponents with the imaginary part equal to zero, i.e. they are sinks and sources without spiralling. On the other hand, if $1/17 < \mu < 1$ then the characteristic exponents have a non-zero imaginary part and we have spiralling sinks and sources. Hence the flow on N_h depends on either $0 < \mu \leqslant 1/17$ (no spirals) or $1/17 < \mu < 1$ (spirals) and it is in both cases gradient like with respect to the v coordinate. These results allow us to visualise the flow which is illustrated in figure 4.

Knowledge of the flow on N_h tells us about orbits that pass close to N_h, i.e. the unbounded orbits of SAM. We see that $\theta(t) \to 0$ for the unbounded orbits and that these orbits oscillate about the $\theta = 0$ axis when $1/17 < \mu < 1$ as mentioned in § 1.

4. Summary

We analysed the unbounded orbits of a swinging Atwood's machine by blowing up the singularity at infinity. The essential idea is to observe that the vector field can have singularities at infinity and that motions in the vicinity of the singularity can be calculated by McGehee's geometric regularisation procedure. This technique is illustrated for the swinging Atwood's machine and allows us to predict the general properties of unbounded trajectories; i.e. the unbounded orbits oscillate about the $\theta = 0$ axis an infinite number of times when $1/17 < \mu < 1$. The general technique illustrated with the swinging Atwood's machine should find application in a wide

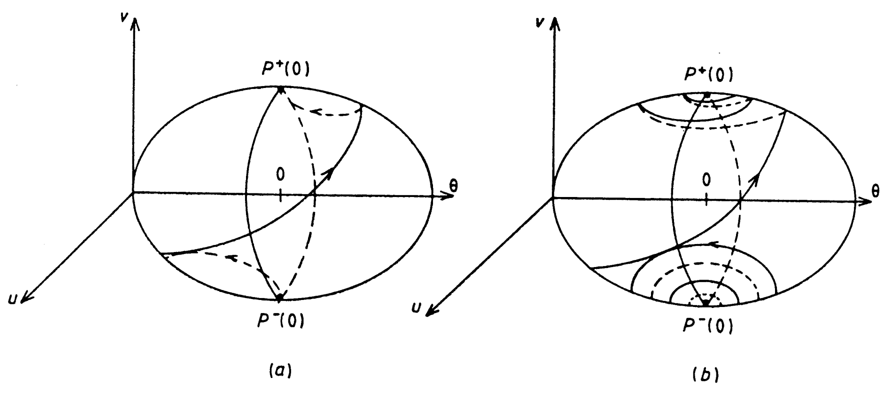

Figure 4. The flow on N_h: (*a*) the case $0 < \mu \leqslant 1/17$; (*b*) the case $1/17 < \mu < 1$.

variety of Hamiltonian systems exhibiting unbounded motions.

References

[1] Bergé P, Pomeau Y and Vidal C 1984 *Order Within Chaos* (New York: Wiley)
Thompson J and Stewart H 1986 *Nonlinear Dynamics and Chaos* (New York: Wiley)

[2] Arnold V I and Avez A 1968 *Ergodic Problems of Classical Mechanics* (Reading, MA: Addison-Wesley)

[3] Moser J 1973 *Stable and Random Motions in Dynamical Systems* (Princeton, NJ: Princeton University Press)

[4] Arnold V I 1978 *Mathematical Methods of Classical Mechanics* (New York: Springer)

[5] Szebehely V 1967 *Theory of Orbits* (New York: Academic) ch 3

[6] McGehee R M 1974 Triple collisions in the collinear three body problem *Invent. Math.* **27** 191

[7] Hirsch M, Pugh C and Shub M 1977 *Invariant Manifolds, Lecture Notes in Mathematics* no 583 (New York: Springer)

[8] Devaney R L 1981 Singularities in classical mechanical systems *Prog. Math.* **10** 221

[9] Devaney R L 1982 Blowing up singularities in classical mechanical systems *Am. Math.* **89** no 8

[10] Lacomba E and Simó C 1982 Boundary manifolds for energy surfaces in celestial mechanics *Celestial Mech.* **28** 37

[11] Tufillaro N B, Abbott T A and Griffiths D J 1984 Swinging Atwood's machine *Am. J. Phys.* **52** 895

[12] Mahnke R and Budde A 1986 *Preprint* Behandlung nichtlinearer Hamiltonscher Systeme auf dem Kleincomputer KC 85/2 *Lehrgrief zur theoretischen Mechanik* WPU Rostock, June

[13] Mahnke R and Budde A 1987 *Preprint* Schwingungsfähiger dynamischer Systeme auf dem Kleincomputer KC 85/2 *Lehrgrief zur theoretischen Mechanik* WPU Rostock, April

[14] Tufillaro N B 1985 Motions of a swinging Atwood's machine *J. Physique* **46** 1495

[15] Tufillaro N B 1986 Integrable motion of a swinging Atwood's machine *Am. J. Phys.* **54** 142

[16] Bruhn B 1987 Chaos and order in weakly coupled systems of nonlinear oscillators *Phys. Scr.* **35** 7

[17] Tufillaro N B 1985 Collision orbits of a swinging Atwood's machine *J. Physique* **46** 2053