[1]The lecture demonstration was observed by H. J. M. at Imperial College, London, in the late 1950s. Presumably, it exists in many other places.
[2]We consider only the case when $E_{12}$ is a constant.
[3]C. Zener, Proc. R. Soc. London Ser. A **137**, 696 (1932).
[4]The reader of Zener's paper should beware that the time-dependent Schrödinger equation has been used with $-i$ instead of $i$ multiplying the time derivative!
[5]The solution in this case is similar to the one we outline for positive $\dot{E}$.
[6]See E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge U.P., London, 1963), 4th ed., pp. 347–349.

# Unbounded orbits of a swinging Atwood's machine

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The motion of a swinging Atwood's machine is examined when the orbits are unbounded. Expressions for the asymptotic behavior of the orbits are derived that exhibit either an infinite number of oscillations or no oscillations, depending only on a critical value of the mass ratio.

## I. INTRODUCTION

A swinging Atwood's machine (SAM)[1] is a simple mechanical system consisting of an ordinary Atwood's machine, in which, however, one of the weights is allowed to swing in a plane (Fig. 1). SAM is a *nonlinear* Hamiltonian system with 2 deg of freedom and hence can display periodic, quasiperiodic, and chaotic behavior.[2] It is also a simple enough system so that it is amenable to analytic, numeric, and experimental studies by undergraduates.[3,4] SAM is integrable[5] when the mass ratio ($\mu = M/m$) is 3, and all the orbits are either periodic or quasiperiodic. For most other values of the mass ratio, SAM can show chaotic behavior,[6,7] a complete analysis of which is still lacking. In fact, the characterization of all motions occurring in 2-deg-of-freedom Hamiltonian systems is still an unsolved problem at the forefront of research in nonlinear dynamics.

In this article, we analyze the unbounded motions of SAM that occur when the mass ratio $\mu \leqslant 1$. Despite the fact that the system is nonlinear, we show that a complete picture of all unbounded motions emerges from an asymptotic analysis of unbounded orbits. In particular, we find that if $\mu \in (1/17, 1]$, then the orbits cross the downward vertical ($\theta = 0$) axis an infinite number of times as $r$ approaches infinity. On the other hand, if $\mu \in (0, 1/17]$, then the orbits do not oscillate about this axis.

## II. HILL'S REGION

The Hamiltonian for SAM in the polar coordinates indicated by Fig. 1 is

$$H(p,q) = \frac{1}{2}\left(\frac{p_r^2}{1+\mu} + \frac{p_\theta^2}{r^2}\right) + gr(\mu - \cos\theta), \qquad (1)$$

where the canonical momenta are $p_r = (1+\mu)\dot{r}, p_\theta = r^2\dot{\theta}$. For convenience, we divide out a factor of $m$ from the energy constant so that the Hamiltonian is solely a function of $\mu$. The Hamiltonian is of the usual type consisting of a kinetic energy term,

$$T = \frac{1}{2}\left(\frac{p_r^2}{1+\mu} + \frac{p_\theta^2}{r^2}\right), \qquad (2)$$

plus the potential energy

$$V = gr(\mu - \cos\theta). \qquad (3)$$

The potential energy for SAM is a homogeneous function of degree 1 and hence the principle of mechanical similarity applies[8]; the orbits on a given energy surface can be rescaled to orbits on other energy surfaces. Therefore, the dynamics generated by SAM are independent of the energy constant. See the Appendix for details.

The fact that the kinetic energy of a Hamiltonian can be positive definite often restricts the motion of a conservative system to a subregion of configuration space, the so-called Hill's region of the mechanical system. Let $h$ denote a particular value for the energy. To find the Hill's region for SAM, observe that the velocity is zero ($p_r = p_\theta = 0$) when

$$R_\mu(\theta) = h/g(\mu - \cos\theta), \qquad (4)$$

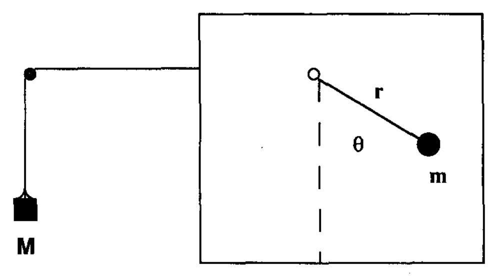


Fig. 1. Swinging Atwood's Machine (SAM). The configuration space variables are the polar coordinates of the swinging bob—$(r,\theta)$. The mass ratio $\mu = M/m$.

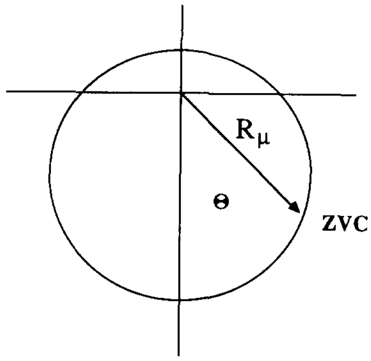


Fig. 2. Hill's region for SAM when $\mu > 1$. Motions are confined to the interior of the zero velocity curve (ZVC).

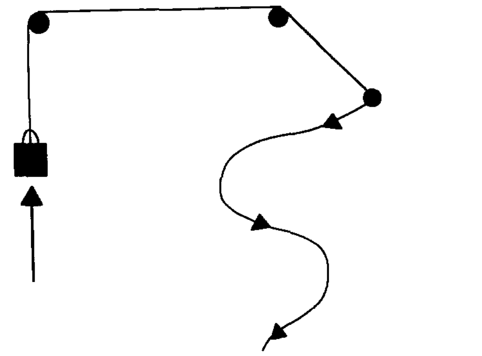

Fig. 4. Unbounded orbit for SAM showing damped oscillations about the downward ($\theta = 0$) axis.

which defines the *zero velocity curve* (ZVC) for SAM. The interior of the ZVC is the Hill's region for SAM. Orbits cannot exist in the exterior of the ZVC because the kinetic energy is positive definite.

The shape of the ZVC depends both on $\mu$ and $h$. For example, suppose $\mu > 1$. Then $gr(\mu - \cos\theta) \geqslant 0$ and, therefore, $h \geqslant 0$. The motion is bounded by $gr(\mu - \cos\theta) \leqslant h$, i.e.,

$$r \leqslant h/g(\mu - \cos\theta).$$

All the orbits are bounded by the ZVC, which in this case is an ellipse with one focus at the origin and eccentricity $1/\mu$ (Fig. 2). Unbounded orbits only exist when $\mu \leqslant 1$.

Consider the Hill's region for $\mu = 1$. As shown in Fig. 3, the motion is bounded in all directions *except* for $\theta = 0$. From (4) it is easy to see that as $R_u(\theta) \to \infty$, $\theta \to 0$ in the ZVC. This means that any unbounded orbit must also have the property that $\theta(t)$ approaches 0 as $r(t)$ goes to infinity. In fact, as illustrated in Fig. 4, numerical solutions show that almost all initial conditions lead to unbounded orbits that oscillate about the $\theta = 0$ axis. This is not so surprising when we consider that the centrifugal pseudoforce of the swinging bob imparts an effective "extra weight" to this mass. It is now easy to see that as $r(t)$ becomes large, the asymptotic motion is simply that of an ordinary Atwood's machine with equal masses traveling at a constant speed,

$$\dot{r}(\infty) = \sqrt{h/m}, \tag{5}$$

where $h$, the energy constant, can be calculated from the initial conditions. The asymptotic value for the radial velocity is easily checked by numerical simulations.

When $\mu < 1$, the energy constant can also be negative. The shape of the Hill's region for $\mu < 1$ is shown in Fig. 5. The region where the motion is unbounded opens up to $\theta \in [-\arccos\mu, \arccos\mu]$ since $gr(\mu - \cos\theta) \leqslant h$ implies $\theta \in [-\arccos(\mu - h/gr), \arccos(\mu - h/gr)]$. From an examination of just Fig. 5, we might conclude that unbounded asymptotic motions could oscillate with arbitrary angular amplitude provided they remain within the Hill's region. However, numerical simulations show that *when* $\mu < 1$, $\theta(t)$ *approaches* 0 *as* $r(t)$ *approaches infinity*. That is, unbounded motions oscillate with a smaller and smaller amplitude even when $\mu < 1$. Moreover, the radial acceleration becomes constant, approaching the Atwood's acceleration given by

$$a = [(\mu - 1)/(\mu + 1)]g. \tag{6}$$

In Sec. III we shall use these facts to study the asymptotic motion in more detail.

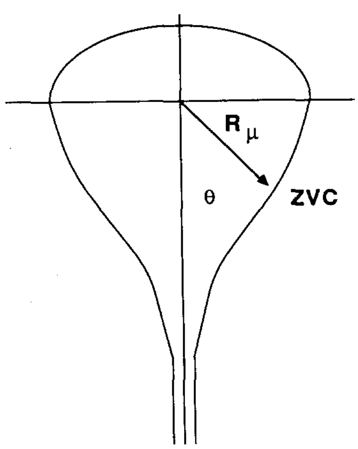


Fig. 3. Hill's region for SAM when $\mu = 1$ and $h > 0$.

## III. UNBOUNDED MOTIONS

From Ref. 1 the equations of motion for SAM are the "radial equation,"

$$(1 + \mu)\ddot{r} = r\dot{\theta}^2 + g(\cos\theta - \mu), \tag{7}$$

and the "angular equation,"

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0. \tag{8}$$

In Sec. II, we saw that the radial component of the unbounded motions of SAM approach that of an ordinary Atwood's machine. This occurs because in the limit as $t \to \infty$, and $r(t) \to \infty$, the angle $\theta(t) \to 0$. Taking this limit

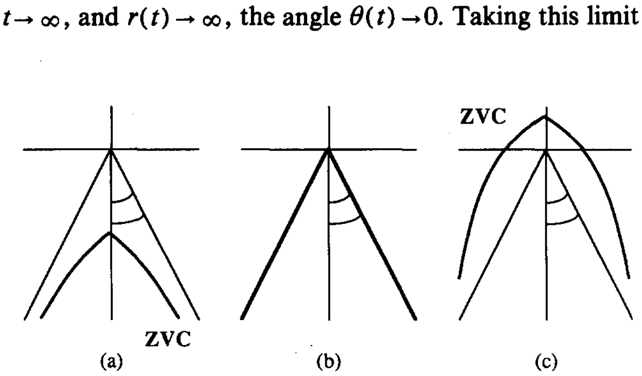


Fig. 5. Hill's region of SAM for $\mu < 1$: (a) case $h < 0$; (b) case $h = 0$; and (c) case $h > 0$. Here, we have used the notation $\sphericalangle = \arccos(\mu)$.

in the radial equation (7) and plugging the resulting Atwood's solution into the angular equation (8) allows us to study the unbounded asymptotic orbits of SAM.

### A. Case $\mu = 1$

Keeping only the first-order terms in $\theta$, and setting $\mu = 1$, we get from (7),

$$\ddot{r} = 0 \Rightarrow r(t) = r_0 + \sqrt{h/m}\, t, \tag{9}$$

where the integration constant is determined from the energy relation. Plugging the asymptotic solution to the radial equation (9) into the angular equation (8) yields

$$(r_0 + \sqrt{h/m}\, t)\ddot{\theta} + 2\sqrt{h/m}\,\dot{\theta} + g\sqrt{h/m}\,\theta = 0, \tag{10}$$

where we assumed $\sin\theta \approx \theta$. Equation (10) can be transformed into a Bessel equation[9] of order 1 by the change of variables,

$$\tau = 2\sqrt{m/h}\,\sqrt{(r_0 + \sqrt{h/m}\, t)g}, \quad \varphi = \theta\sqrt{r_0 + \sqrt{h/m}\, t} \tag{11}$$

resulting in

$$\tau^2 \frac{d^2\varphi}{d\tau^2} + \tau \frac{d\varphi}{d\tau} + (\tau^2 - 1)\varphi = 0. \tag{12}$$

The properties of the Bessel equation (12) can be translated into the properties of the angular motion of the swinging bob by examining

$$\theta(t) = \frac{1}{\sqrt{r_0 + \sqrt{h/m}\, t}}\,\varphi \times\left[2\sqrt{\frac{m}{h}}\sqrt{\left(r_0 + \sqrt{\frac{h}{m}}\,t\right)g}\,\right]. \tag{13}$$

To uncover the qualitative properties of the Bessel equation—and hence the angular equation—consider the additional change of variables[10] applied to (12),

$$\varphi = u/\sqrt{\tau} \tag{14}$$

leading to

$$u'' + (1 - 3/4\tau^2)\, u = 0, \tag{15}$$

which for large $\tau$ becomes approximately

$$u'' + u = 0, \tag{16}$$

with solution $u = A\cos(\tau + B)$, i.e.,

$$\varphi(\tau) = (A/\sqrt{\tau})\cos(\tau + B), \tag{17}$$

where $A$ and $B$ are constants.

From (17) and (13), we can now predict the asymptotic properties of the unbounded orbits when $\mu = 1$. Specifically, we see that the angle $\theta(t)$ decays sinusoidally to zero, and that the magnitudes of its extrema decrease like $(r_0 + \sqrt{h/m}\, t)^{-3/4}$. Despite the fact that the angle decreases, the sweep defined by $s \equiv r\theta = (r_0 + \sqrt{h/m}\, t)\theta(t)$ grows sinusoidally in time with the magnitude of its extrema monotonically increasing like $(r_0 + \sqrt{h/m}\, t)^{1/4}$. Both of these predictions are easy to check and observe by numerical simulations.

### B. Case $\mu < 1$

In this case, the small angle limit of (7) results in

$$(1 + \mu)\ddot{r} = (1 - \mu)g \Rightarrow r(t) = r_0 + \tfrac{1}{2}at^2,$$

where $a$ is the Atwood's acceleration defined by Eq. (6). If by convention we agree to take $t_0 = \sqrt{2r_0/a} > 0$, then the Atwood's solution can be reexpressed as

$$r(t) = \tfrac{1}{2}at^2, \quad t_0 > 0. \tag{18}$$

To study the asymptotic behavior of the unbounded motions, we plug the Atwood's solution (18) into the angular equation (8) of SAM. The result, after a little algebra, is

$$\ddot{\theta} + \frac{4}{t}\,\dot{\theta} + \frac{2}{t^2}\left(\frac{1+\mu}{1-\mu}\right)\theta = 0, \tag{19}$$

where again we assumed $\sin\theta \approx \theta$. Equation (19) is easily solved with the transformation $\tau = \ln t$ yielding

$$\frac{d^2\theta}{d\tau^2} + 3\,\frac{d\theta}{d\tau} + 2\left(\frac{1+\mu}{1-\mu}\right)\theta = 0, \tag{20}$$

a linear equation with constant coefficients. There are three possible motions depending on whether $\mu > 1/17$, $\mu = 1/17$, or $\mu < 1/17$. In the last two cases, it is easy to show that the bob is falling too quickly for there to be oscillations about the $\theta = 0$ axis.

Damped oscillations of the form

$$\theta(t) = Ct^{-3/2}\cos(\alpha \ln t + D),$$
$$\alpha = \sqrt{(17\mu - 1)/4(1 - \mu)} \tag{21}$$

occur when $\mu > 1/17$. Here, $C$ and $D$ are constants determined by the initial data. This is similar to the $\mu = 1$ case except that the angle decays toward zero like $t^{-3/2}$, and the sweep increases according to $t^{1/2}$.

## IV. SUMMARY

We analyzed the unbounded orbits of a swinging Atwood's machine that occur when the mass ratio $\mu \leqslant 1$. If $\mu \in [1, 1/17)$, then these orbits undergo damped oscillations about the $\theta = 0$ axis with decay rates predicted from an asymptotic analysis of the equations of motion for a swinging Atwood's machine. If $\mu \in (0, 1/17]$, then the angular variable also decays toward zero without crossing the $\theta = 0$ axis, except possibly in the nonasymptotic regime. Our analysis provides a rather complete picture for the unbounded orbits of a swinging Atwood's machine.


## ACKNOWLEDGMENTS

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## APPENDIX: RESCALED SAM HAMILTONIAN

Studying the general dynamics that arise from SAM can be simplified somewhat by rescaling the Hamiltonian. For instance, the energy can be rescaled by the similarity transformation,[8]

$$\bar{r} = h_r, \quad \bar{t} = h^{1/2}t, \quad \bar{p}_r = h^{1/2}p_r, \quad \bar{p}_\theta = h^{-1/2}p_\theta,$$

which carries the Hamiltonian (1) from $H = h$ to $H = 1$ or $H = -1$ according to $h > 0$ and $h < 0$, respectively. Therefore, it is sufficient to study the energy levels $H = 1$, $H = 0$, and $H = -1$. In addition, the gravity constant can be rescaled by

$$\bar{g} = hg, \quad \bar{t} = h^{-1/2}t, \quad \bar{p}_r = h^{-1/2}p_r, \quad \bar{p}_\theta = h^{-1/2}p_\theta,$$

so we can set $g = 1$ without loss of generality.

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[1]N. B. Tufillaro, T. A. Abbott, and D. J. Griffiths, Am. J. Phys. **52**, 895 (1984).
[2]For an elementary introduction to the new developments in nonlinear dynamics, see P. Bergé, Y. Pomeau, and C. Vidal, *Order Within Chaos* (Wiley, New York, 1984); or J. Thompson and H. Stewart, *Nonlinear Dynamics and Chaos* (Wiley, New York, 1986).
[3]R. Mahnke and A. Bundde: "Behandlung Nichtlinearer Hamiltonscher Systeme auf dem Kleincomputer KC 85/2," Lehrbrief zur theoretischen Mechanik, WPU Rostock, Juni 1986.
[4]R. Mahnke and A. Bundde: "Analyse Schwingungsfähiger Dynamischer Systeme auf dem Kleincomputer KC 85/2," Lehrbrief zur theoretischen Mechanik, WPU Rostock, April 1987.
[5]N. B. Tufillaro, Am. J. Phys. **54**, 142 (1986).
[6]N. B. Tufillaro, J. Phys. **46**, 1495 (1985).
[7]B. Bruhn, Phys. Scripta **35**, 7 (1987).
[8]See L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon, Oxford, 1976), 3rd ed., Sec. 10.
[9]R. Borrelli, C. Coleman, and D. Hobson, Math. Mag. **58**, 79 (1985).
[10]G. Birkhoff and G. Rota, *Ordinary Differential Equations* (Blaisdell, Waltham, MA, 1969), 2nd ed., p. 302.

# The effect of the mass of the center spring in one-dimensional coupled harmonic oscillators

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The effect of the mass of the center spring in the one-dimensional coupled harmonic oscillator is calculated and shown to have a small but noticeable effect on the periods of the normal modes of oscillation. Sophomore engineering students have developed the experimental technique using only stop watches, an analytic balance, and a standard Ealing™ air track.

## I. INTRODUCTION

The mechanical coupled harmonic oscillator (Fig. 1) consisting of three springs alternating with two masses in a straight line, stretched between two fixed walls, and moving in the direction of the line without friction, has long been a student's early introduction to normal modes and coupled linear differential equations.[1,2] An early experiment in a student's experience is often simple harmonic motion using a mass suspended vertically by a spring. In laboratory manuals, the correction for the spring's mass is either ignored[3] or suggested without proof to be $\frac{1}{3}$ the mass of the spring.[4] It is just as well because the value $\frac{1}{3}$ is only applied to the cylindrical spring (the proof is typically left for exercises in texts[5,6]) since there is still disagreement in the literature for the tapered spring.[6–10] The proof of the $\frac{1}{3}$ value will be reviewed so that the method can be extended to the middle spring. Finally the periods of the normal modes will be calculated and demonstrated using an air track.

## II. REVIEW OF THE EFFECT OF A SPRING FIXED AT ONE END

Consider the system (Fig. 2) of a simple harmonic oscillator consisting of a block of mass $M$ connected to one end of a spring of mass $m$ and stiffness (Hooke's constant) $k$. The frictionless motion is confined to one horizontal dimension, and the other end of the spring is fixed. The neutral length of the spring is $L$, and the displacement $(x)$ of the block is measured from the location of the block when the spring is neutral. The kinetic energy of the block is $\frac{1}{2}M\dot{x}^2$ and the potential energy of the spring is $\frac{1}{2}kx^2$. The kinetic energy of the spring is the integral of the kinetic energy of each mass element $dm$ of the spring. It is assumed that the spring is uniformly stretched[11] so that the linear mass density $\sigma = dm/dz = m/Z$, where $\sigma$ is uniform at any instant and $Z = x + L$. It is further assumed that the velocity of $dm$ is $\dot{z}$ such that

$$\dot{z}/z = \dot{x}/Z. \qquad (1)$$

Hence, the spring's kinetic energy becomes

$$K_{sp} = \frac{1}{2}\int_0^m \dot{z}^2\, dm = \left(\frac{\sigma \dot{x}^2}{2Z^2}\right)\int_0^Z z^2\, dz = \frac{1}{2}\frac{m}{3}\dot{x}^2. \qquad (2)$$

The total energy of the system is

$$E = \tfrac{1}{2}(M + m/3)\dot{x}^2 + \tfrac{1}{2}kx^2. \qquad (3)$$

Since the energy is constant,

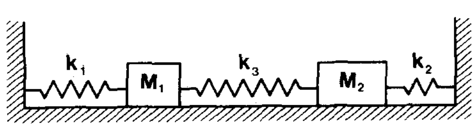


Fig. 1. The one-dimensional mechanical coupled harmonic oscillator.

  