# Integrable motion of a swinging Atwood's machine

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The motion of a swinging Atwood's machine is shown to be integrable when the mass ratio is three. Hamilton–Jacobi theory is used to explicitly exhibit the first integral.

## I. INTRODUCTION

Arnold writes[1]

The collection of solvable "integrable" problems which we have at our disposal is not large (one-dimensional problems, motion of a point in a central field, eulerian and lagrangian motions of a rigid body, the problem of two fixed centers, and motion along geodesics on the ellipsoid). However, with the help of these "integrable cases," we can obtain meaningful information about many important systems by considering an integrable problem as a first approximation.

A more extensive set of soluble problems in particle dynamics is presented by Whittaker.[2] Many recent discoveries such as the Toda Lattice could also be added to this list.

We shall add another hidden treasure to this collection by showing that the motion of a "swinging Atwood's machine" (SAM) is integrable when the ratio of nonswinging mass $M$ to the swinging mass $m$ is three. SAM (Fig. 1) is an ordinary Atwood's machine, in which, however, one of the weights is allowed to swing in a plane. Many of the elementary properties of SAM are discussed in a recent article in this journal.[3]

In this paper Hamilton–Jacobi theory will be employed to prove integrability when the mass ratio is three. Since SAM has two-degrees of freedom, it will suffice to find a single first integral in addition to the total energy. This constant of the motion will arise as one of the separation constants in the corresponding Hamilton–Jacobi equation. It is totally nonobvious and this example provides a lovely illustration of the Hamilton–Jacobi method in solving concrete problems which cannot be handled by more elementary techniques.

Hamilton–Jacobi theory is discussed in graduate courses in classical mechanics.[1,4] However, it is often not stressed that for most systems it is not possible to discover just the right canonical transformation which leads to an exact solution.[5] This fact has been known since the time of Poincaré (1982) who, along with Bruns, showed that in general, most classical mechanics problems are not integrable, i.e., there do not exist $N$ (the degrees of freedom) single-valued analytic time-independent functions on phase space satisfying $dI/dt = 0$.[6] Nevertheless, a surprising number of problems which arise in physics do turn out to be integrable and even separable in some coordinate system.

## II. HAMILTON–JACOBI METHOD

The kinetic energy for SAM is

$$T = \tfrac{1}{2}(m+M)\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2, \qquad (1)$$

where $r$ and $\theta$ are specified in Fig. 1. The potential energy is

$$V = gr(M + m\cos\theta), \qquad (2)$$

where $g$ is the acceleration due to gravity. The total energy is

$$E = T + V = \tfrac{1}{2}(m+M)\dot{r}^2 + \tfrac{1}{2}mr^2\dot{\theta}^2 + gr(M - m\cos\theta). \qquad (3)$$

The corresponding Hamiltonian, obtained with the generalized momentum ($p_i = \partial L/\partial\dot{q}_i$, $L = T - V$):

$$p_r = (m+M)\dot{r}, \qquad (4a)$$

$$p_\theta = mr^2\dot{\theta}, \qquad (4b)$$

is

$$H(p,q) = \frac{1}{2}\left(\frac{p_r^2}{m+M} + \frac{p_\theta^2}{mr^2}\right) + gr(M - m\cos\theta). \qquad (5)$$

Now the game of Hamilton–Jacobi theory is to find some canonical tranformation which will separate variables in the Hamilton–Jacobi equation. A fortuitous choice for the SAM Hamiltonian (5) is

$$r = \tfrac{1}{2}(\xi^2 + \eta^2), \qquad (6a)$$

$$\theta = 2\arctan[(\xi^2 - \eta^2)/2\xi\eta], \qquad (6b)$$

which is a coordinate transformation from polar coordinates to parabolic cylinder coordinates plus a certain stretch in the angular variable. The inverse transformation of (6) is

$$\xi^2 = r[\pm 1 + \sin(\theta/2)], \qquad (7a)$$

$$\eta^2 = r[\pm 1 - \sin(\theta/2)]. \qquad (7b)$$

The total energy in the new coordinates [apply tranformation (6) to (3)] is

$$E = [\tfrac{1}{2}(m+M)\xi^2 + 2m\eta^2]\dot{\xi}^2 + [\tfrac{1}{2}(m+M)\eta^2 + 2m\xi^2]\dot{\eta}^2 + (M-3m)\xi\eta\dot{\xi}\dot{\eta} + (g/2)(\xi^2+\eta^2)^{-1} \times [(M+m)(\xi^4+\eta^4) + (M-3m)\xi^2\eta^2], \qquad (8)$$

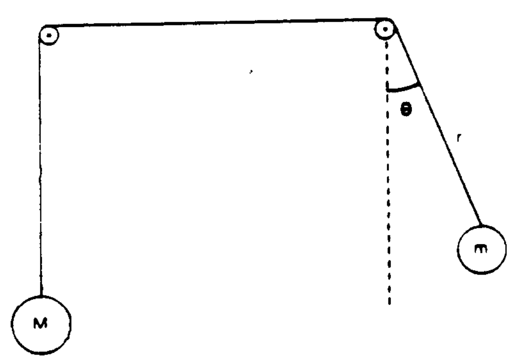


Fig. 1. SAM. Swinging Atwood's machine. We consider the case where $M = 3m$.

which simplifies if we set $M = 3m$, as is done throughout the rest of this paper. In parabolic cylinder coordinates the generalized momentum is $(M = 3m)$,

$$p_\xi = 4\dot{\xi}(\xi^2 + \eta^2), \tag{9a}$$

$$p_\eta = 4\dot{\eta}(\xi^2 + \eta^2), \tag{9b}$$

from which we calculate the Hamiltonian

$$H(p,q) = [1/(\xi^2 + \eta^2)] \times [\tfrac{1}{8}(p_\xi^2 + p_\eta^2) + 2g(\xi^4 + \eta^4)] = E. \tag{10}$$

The Hamilton–Jacobi equation formed from Hamiltonian (10) is

$$\left(\frac{\partial S}{\partial \xi}\right)^2 + \left(\frac{\partial S}{\partial \eta}\right)^2 + 16g(\xi^4 + \eta^4) = 8E(\xi^2 + \eta^2), \tag{11}$$

where $S$ is the generating function. The variables $\xi$ and $\eta$ are clearly separated in the Hamilton–Jacobi equation (11), so let the generating function be expressed as the sum

$$S(\xi,\eta) = S_\xi(\xi) + S_\eta(\eta). \tag{12}$$

Then on separating variables we find

$$\left(\frac{\partial S}{\partial \xi}\right)^2 + 16g\xi^4 - 8E\xi^2 = I, \tag{13a}$$

$$-\left(\frac{\partial S}{\partial \eta}\right)^2 - 16g\eta^4 + 8E\eta^2 = I, \tag{13b}$$

where $I$ is the separation constant. Solving for the generating function $S$ in Eqs. (13) gives the complete solution to the Hamilton–Jacobi equation (11) as

$$S(\xi,\eta,I) = \int (I + 8E\xi^2 - 16g\xi^4)^{1/2}\, d\xi + \int (-I + 8E\eta^2 - 16g\eta^4)^{1/2}\, d\eta. \tag{14}$$

Although (14) is exactly soluble in terms of elliptic integrals,[7] we would instead like to turn our attention to deriving an explicit form for the other constant of the motion $I$. To this end add (13a) and (13b) and use (9) and (10) to eliminate the energy $E$. In this way we get the first integral

$$I = 2(\xi^2 + \eta^2)(\eta^2\dot{\xi}^2 - \xi^2\dot{\eta}^2) + 2g\xi^2\eta^2[(\xi^2 - \eta^2)/(\xi^2 + \eta^2)]. \tag{15}$$

In order to arrive at the first integral in our original coordinate system we apply the inverse transformation (7) to (15). The first integral in polar coordinates reads

$$I(r,\dot{r},\theta,\dot{\theta}) = r^2\dot{\theta}\left[\dot{r}\cos\left(\frac{\theta}{2}\right) - \frac{r\dot{\theta}}{2}\sin\left(\frac{\theta}{2}\right)\right] + gr^2\sin\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right). \tag{16}$$

A very hidden symmetry!

## III. DISCUSSION

We originally began to suspect that the motion was integrable for $M/m = 3$ because of the singular behavior described in Ref. 3 and numerical evidence.[8] These same clues suggest the motion is integrable when

$$M/m = 4n^2 - 1 = 3,15,35,\ldots,n \in \mathbf{Z}.$$

The case of $n = 1$ has been proven. The first integral (16) turns out to be quadratic in the velocity terms. For $n = 2$ we have demonstrated that if a first integral exists, then it is not quadratic in velocities, but at least cubic. This proof is based on work by Ankiewicz and Pask.[9]


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[1]V. I. Arnold, *Mathematical Methods of Classical Mechanics* (Springer, New York, 1978).

[2]E. T. Whittaker, *A Treatise on Analytical Dynamics of Particles and Rigid Bodies* (Cambridge U. P., Cambridge, 1927).

[3]N. B. Tufillaro, T. A. Abbott, and D. J. Griffiths, Am. J. Phys. **52**, 895 (1984).

[4]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1980).

[5]A. J. Dragt and J. M. Finn, J. Geophys. Res. **81**, 2327 (1976).

[6]H. Poincaré, *Les Methodes Nouvelles de la Mechanique Celests* (reprinted by Dover, New York, 1957).

[7]L. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products* (Academic, New York, 1980), e.g., 2.290.

[8]N. B. Tufillaro, "Regular and Chaotic Motion of a Swinging Atwood's Machine," J. Phys. (Paris) **46**, 1495 (1985).

[9]A. Ankiewicz and C. Pask, J. Phys. A Math. Gen. **16**, 4203 (1983).