Smiles

and

Teardrops

A Thesis

Presented to the

Division of Mathematics and Natural Science

Reed College

In Partial Fulfillment

of the Requirements for the Degree

Bachelor of Arts

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by

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April 1982

ABSTRACT

An Atwood's machine in which one of the masses swings like a pendulum is a simple realization of a conservative, oscillatory system with two degrees of freedom. A study of this system is undertaken here by means of: (1) numerical solution to the equations of motion, (2) perturbative solutions, and (3) a laboratory model. The numerical studies indicate the existence of a wide variety of periodic motions. Perturbative solutions to the so-called Smile and Teardrop trajectories are compared to their numerical counterparts; the agreement is surprisingly good. Progress toward understanding the global structure of the periodic motions is also reported.

Keywords: Dynamical System, Differential Equation, Coupled Pendulum

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CONTENTS

"Dynamical systems with two degrees of freedom constitute the simplest type of non-integrable dynamical problems and possess a very high degree of mathematical interest."

G.D. Birkhoff, 1917

"Analyzing a general potential system with two degrees of freedom is beyond the capability of modern science."

V.I. Arnold, 1974

I. Coupled Oscillators

Two pendula can be connected together in a variety of ways to produce a coupled oscillator. A simple toy is made by connecting a pair of pendula with a drinking straw (see Fig. 1-1).

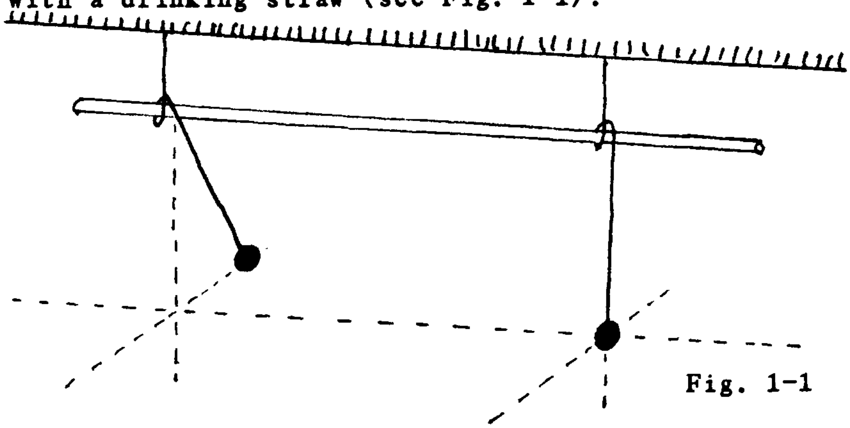
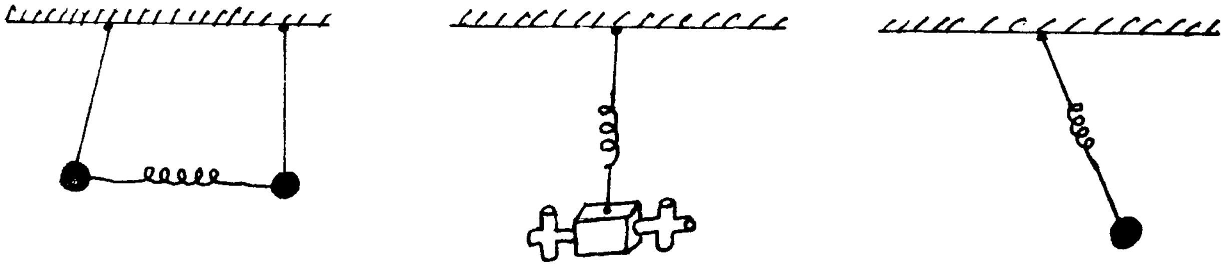


Fig. 1-1

Drawing one of the bobs aside and then releasing it gives rise to motion which is characteristic of coupled oscillators. The pendulum that was initially at rest gradually begins to oscillate until both bobs are swinging with equal amplitudes. As the process continues, the bob that was first displaced eventually comes to rest. The starting condition is now reversed and the pattern will repeat itself. The energy in the swinging bob is transmitted through the straw to the other bob. In this way, energy is shuttled back and forth between the two oscillators.

A number of coupled pendulum systems have been the subject of detailed quantitative study. Most notable in this regard are the Spring-coupled pendulum, the Wilberforce pendulum, and the Elastic pendulum.



Spring-coupled Wilberforce Elastic

Fig. 2-1

The promising young physicist is hypnotized by the motions of these simple machines. The trajectories can be absurdly complex or delightfully simple. These three systems vary considerably in the degree to which their possible motions have been analyzed. For instance, any motion of the Spring-coupled pendulum can be described as a superposition of normal modes; the problem is linear and completely solved.* The Elastic pendulum, on the other hand, poses an essentially nonlinear problem. Periodic motions are known to exist, and a great deal of analytic and experimental work has been done; but a complete understanding of the system is still lacking. Active research into the elastic pendulum is still under way.†

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* For a detailed discussion of the Spring-coupled and Wilberforce pendulum, see: A.P French, Vibrations and Waves (Norton, New York 1971) Chap. 5.

† A recent study is by E. Breitenberger and R.D. Mueller, "The elastic pendulum: A nonlinear paradigm", J. Math. Phys. 22, 6 (1981).

II. SAM

To this class of machines, I would like to introduce the Swinging Atwood's Machine; hereafter known as SAM.† The system is an Atwood's machine in which one of the masses is allowed to swing. SAM is a hybrid of a simple pendulum and an Atwood's machine.

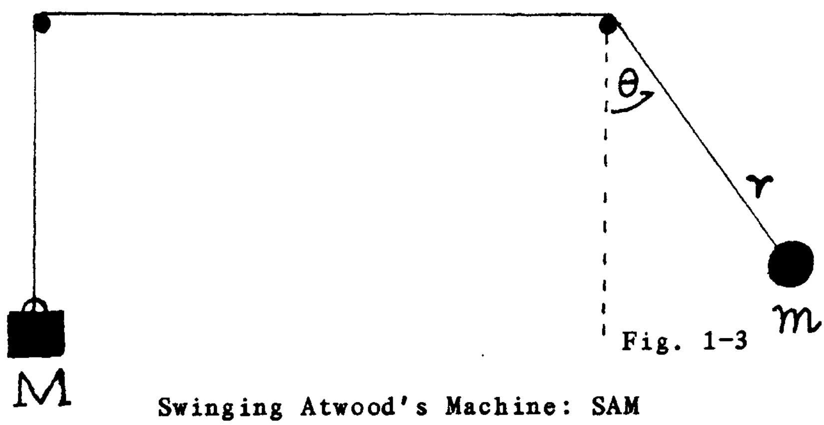


Fig. 1-3

Swinging Atwood's Machine: SAM

The device is constructed from two masses, M and m, which are connected by a weightless, perfectly flexible and inextensible cord. The masses are supported by two frictionless and weightless pulleys. The bob, m, is free to move in the plane. The block, M, is constrained to move in the vertical direction only, up and down, in one dimension. There are no dissipative forces in the system so the total energy is constant.

The problem is to describe the trajectories of the swinging mass. For example, what happens if the swinging mass is drawn aside and then released (see Fig. 1-3)? Suppose, for a moment, that m is much heavier than M: then m will swing down, away from the pulley. But what happens when m is a little lighter than M? Initially, it swings in toward the pulley since M is heavier. As m begins to swing, though, it picks up a

† An effort is made to use TLA's throughout (Three Letter Acronyms).

centrifugal pseudo-force which throws it outward. However, the centrifugal force diminishes as m gets further from the pulley. So it can't go out too far before it heads back in again. Of course, this analysis is quite naive and intuitive; but it does lead one to suspect that there may be motions which are confined to an annular region neither too close nor too far from the pulley.

SAM is as simple to describe as any of the coupled pendulum machines before it, yet its motions turn out to be astonishingly complex. It is an essentially nonlinear problem insofar as the interesting motions are not predicted by the linear theory, and are the result of nonlinear coupling. The problem is difficult since it is equivalent to describing the motions of a particle with two degrees of freedom under a specific noncentral force.

III. Summary of Chapters

Smiles and Teardrops is a state of the art account of SAM. In Chapter I the equations of motion are derived. Chapter II opens with some typical numerical solutions. These computer studies illustrate the basic properties of the system. They suggest the existence of a wide spectrum of periodic trajectories and also cultivate one's intuition.

The remaining chapters concentrate on periodic solutions. A time-dependent perturbation scheme gives excellent fits for the so-called Smile and Teardrop trajectories. Smiles (studied in Chapter IV) are periodic trajectories which derive from the motion of a simple pendulum. Chapter V deals with solutions that are close to those of an Atwood's machine. The simplest solution of this type looks like a teardrop. Finally, Chapter VI features a formal argument demonstrating the existence of periodic trajectories.

The last chapter contains a grab-bag of observations, examples, and comments that could be useful in an undergraduate mechanics course.

IV. How to build SAM*

SAM can be constructed from standard air table equipment. I used apparatus purchased from the Ealing Corporation. The necessary ingredients are:

1. Large air table (4 x 4 ft)
2. Air blower
3. Center table pulley
4. General purpose air pulley
5. Fine steel cable
6. Mylar recording tape
7. Puck (50 grams)
8. Weights

Figure 1-4 shows how the parts are assembled.

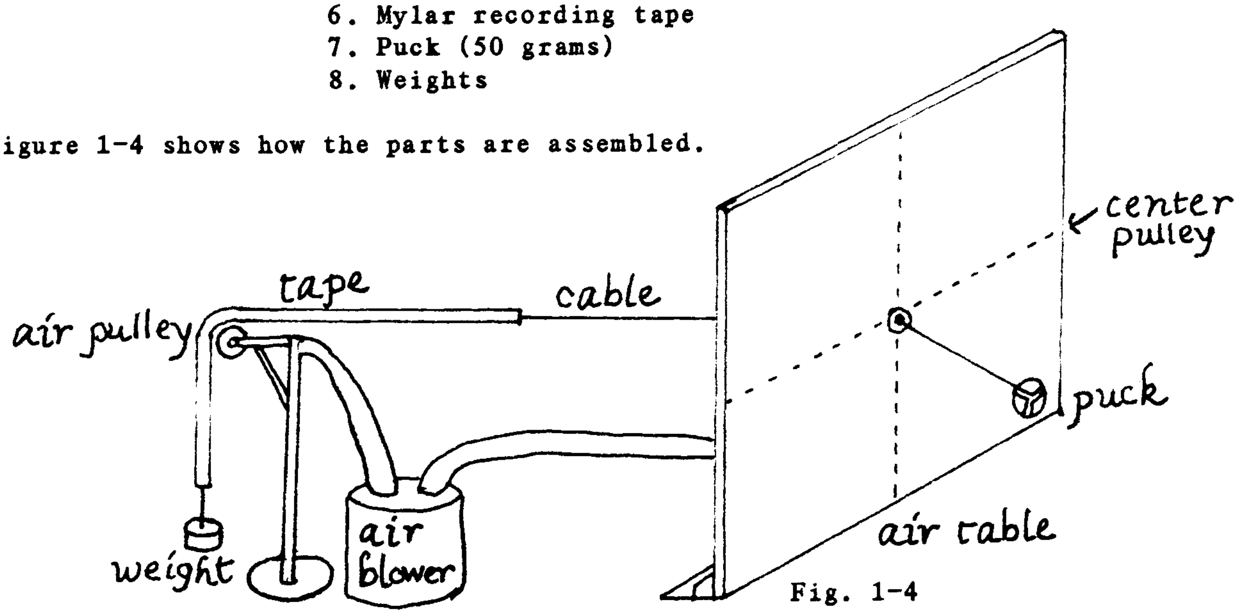


Fig. 1-4

The puck constitutes the swinging mass; it glides on the air table which helps confine it to the vertical plane. A steel cable connects the puck to a piece of ordinary mylar recording tape which slides across a frictionless air pulley. The tape is connected to the nonswinging mass, an adjustable weight. The steel cable passes through a center table pulley which allows free rotation about the horizontal axis.

* Patent pending

CHAPTER II. The Equations

"The beauty and almost divine simplicity of these equations is such that these formulae are worthy to rank with those mysterious symbols which in ancient times were held directly to indicate the Supreme Reason at the base of all things."

A.N. Whitehead

"You can find Lagrange's equations for almost anything; you can solve them for almost nothing."

Nick

I. The Equations of Motion

An elegant and useful formulation of classical mechanics is provided by the Lagrange equations,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0, \quad j = 1,2,\ldots,n. \tag{2-1}$$

where q_j are the generalized coordinates, and the Lagrangian, L, is the difference between the kinetic and potential energy:

$$L = T - V. \tag{2-2}$$

Polar coordinates are a natural choice for marking the position of the swinging mass, with the angular displacement measured counterclockwise from the plumb, as shown in Fig. 1-3. Call m the mass of the swinging pendulum bob and let M represent the other bob. As usual, g denotes the gravitational acceleration. Then the kinetic energy is

$$T = \frac{1}{2}M\dot{r}^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2), \tag{2-3}$$

while the potential energy is

$$V = gr(M - m\cos\theta), \tag{2-4}$$

except for a constant which is omitted. Combining the two yields

$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}M\dot{r}^2 + gr(m\cos\theta - M). \tag{2-5}$$

Let $r = q_1$ and $\theta = q_2$, then Lagrange's equations become

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta},$$

which result in

$$(m+M)\ddot{r} = mr\dot{\theta}^2 + g(m\cos\theta - M) \tag{2-6}$$

$$\frac{d}{dt}(mr^2\dot{\theta}) = -mgr\sin\theta. \tag{2-7}$$

These two equations admit the following Newtonian interpretation. Equation (2-6) expresses Newton's second law in the radial coordinate: the total mass times the acceleration in the radial direction, $(m+M)\ddot{r}$, is equal to the sum of the radial forces; the centrifugal pseudo-force, $mr\dot{\theta}^2$, and the radial component of the gravitational force, $g(m\cos\theta - M)$. Equation (2-7) is the same as for a simple pendulum, except in this instance $\dot{r} \neq 0$. The term $mr^2\dot{\theta}$ is the angular momentum while $-mgr\sin\theta$ is the torque. So this equation says that the time rate of change of angular momentum is equal to the applied torque,

$$\frac{d\vec{L}}{dt} = \vec{N},$$

where $\vec{L} = \vec{r} \times \vec{p}$ and $\vec{N} = \vec{r} \times \vec{F}$. In practice, the equations of motion are derived more swiftly from Newtonian first principles than the Lagrangian formalism.

To simplify matters, define

$$\mu = \frac{M}{m}.$$

Notice that

$$\mu < 1, \text{ if } M < m;$$

$$\mu = 1, \text{ if } M = m;$$

$$\mu > 1, \text{ if } M > m.$$

Equations (2-6) and (2-7) now take on their final form:

$$(1 + \mu)\ddot{r} = r\dot{\theta}^2 + g(\cos\theta - \mu) \qquad (2\text{-}8)$$

$$\frac{d}{dt}(r^2\dot{\theta}) = -gr\sin\theta \quad \text{or, differentiating:} \qquad (2\text{-}9)$$

$$\ddot{\theta} + 2\frac{\dot{r}}{r}\,\dot{\theta} + \frac{g}{r}\sin\theta = 0. \qquad (2\text{-}10)$$

The total energy (T+V) is constant, since the system is conservative. For convenience divide out a factor of m and define $E = \frac{1}{m}(T+V)$:

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \frac{1}{2}\mu\dot{r}^2 + gr(\mu - \cos\theta). \qquad (2\text{-}11)$$

It is easy to verify, by differentiating (2-8) and (2-9), that it is indeed a constant of the motion.

SAM's motion is completely determined by the pair of second order nonlinear coupled ordinary differential equations represented by (2-9) and (2-8).* Equation (2-8) is hereafter referred to as the <u>radial equation</u>, while equation (2-9) or (2-10) is known as the <u>angular equation</u>. Equations (2-8) through (2-11) serve as the starting point for all

* William E. Asher, "R.U. Experienced?", Reed College Chemistry Thesis (1980) See Reference 47.

future analysis of SAM's dynamics.

II. Initial Conditions

SAM's motions or trajectories are uniquely determined when the initial conditions are applied to the angular and radial equations. Notation for the most general set of initial conditions is established as follows:

i. General initial conditions

$$r(0) = r_0 \qquad \theta(0) = \theta_0$$
$$\dot{r}(0) = \dot{r}_0 \qquad \dot{\theta}(0) = \dot{\theta}_0 .$$

Motions resulting from two particular types of initial conditions will be studied in detail. First, consider those motions that start with the system at rest. Imagine holding the swinging bob in some position and simply letting go; then at the outset the velocity is zero.

ii. Rest initial conditions

$$r(0) = r_0 \qquad \theta(0) = \theta_0$$
$$\dot{r}(0) = 0 \qquad \dot{\theta}(0) = 0 .$$

Second, envision motion that is outwardbound from the origin; these trajectories begin at the center and are fired out radially.

iii. Outwardbound initial conditions

$$r(0) = 0 \qquad \theta(0) = \theta_0$$
$$\dot{r}(0) = \dot{r}_0 \qquad \dot{\theta}(0) = 0 .$$

These two types of initial conditions are of interest since Smiles are examples of solutions that start from rest; Teardrops are outwardbound.

III. Useful Observations

This section contains a few general results that follow immediately from the equations of motion.

The first observation is that there exists a lower bound for $\ddot{r}$. In fact, it must be $-g$ since nothing in the problem can give M a downward acceleration greater than g. Formally:

THEOREM 2.1

$$\ddot{r} \geq -g$$

Proof:

$$\ddot{r} = \frac{1}{(1 + \mu)}[r\dot{\theta}^2 + g(\cos\theta - \mu)] \geq g\frac{\cos\theta - \mu}{(1 + \mu)} \geq -g. \text{ QED}$$

It may come as a surprise that the string is taut, and the tension in the string is positive, even if the swinging mass is above the horizontal.

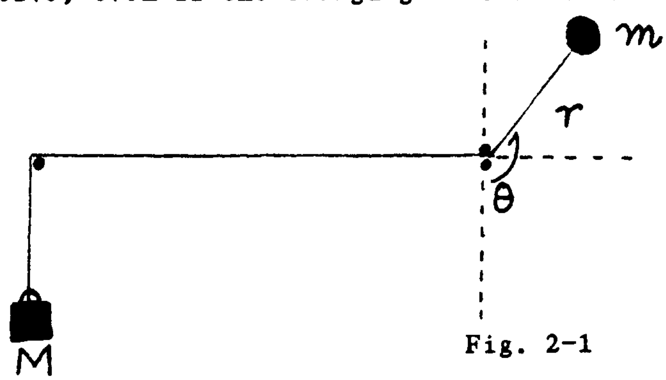


Fig. 2-1

Without the string, m and M are freely falling bodies with the same acceleration, g. Putting the string back into the picture, it is clear that the acceleration of m toward the pulley is less than or equal to g. The tension is zero only when m is directly above the pulley so that the centrifugal force, which is always outward, is zero.

A second interesting fact is that the equations of motion scale in the radial coordinate. Given any trajectory, there exists a geometrically similar trajectory obtained by either expanding or contracting r by a fixed factor. Exactly how this scaling works is shown in the following theorem.

THEOREM 2.2

If $\{r,\theta\}$ is a solution, so is $\{kr,\theta\}$; though the new trajectory is traversed in a different time.

Proof:

The transformation

$$\bar{r} = k\,r$$
$$\bar{t} = k^{\frac{1}{2}}\,t$$

results in

$$\frac{d\bar{r}}{d\bar{t}} = k^{\frac{1}{2}}\,\dot{r}; \qquad \frac{d^2\bar{r}}{d\bar{t}^2} = \ddot{r};$$

$$\frac{d\theta}{d\bar{t}} = \frac{\dot{\theta}}{k^{\frac{1}{2}}}; \qquad \frac{d^2\theta}{d\bar{t}^2} = \frac{\ddot{\theta}}{k},$$

and leaves the radial (2-8) and the angular (2-9) equations unchanged.

QED

The gravitational constant can also be scaled by the transformation

$$\bar{g} = kg$$
$$\bar{t} = \frac{t}{k^{\frac{1}{2}}},$$

so one can set $g = 1$ without loss of generality.

Furthermore, the equations of motion are invariant under time reversal because the potential energy is independent of time. Switching parity of the θ also results in a new solution, since L is an even function in θ.

THEOREM 2.3

If $\{r(t), \theta(t)\}$ is a motion, then $\{r(-t), \theta(-t)\}$ and $\{r(t), -\theta(t)\}$ are also possible motions.

IV. Exact Solutions

Three closed-form solutions are known that satisfy the angular and radial equations exactly. An obvious one is the Atwood solution. If $\theta(t)=0$, then SAM is simply an Atwood's machine.

i. Atwood solution

For $\theta(t) = 0$, $\quad r(t) = r_0 + \dot{r}_0 t + \frac{1}{2}g\,\frac{1-\mu}{1+\mu}\,t^2.$

For $\theta(t) = \pi$, $\quad r(t) = r_0 + \dot{r}_0 t - \frac{1}{2}g\,t^2.$

The projectile solution is a bit more complicated to verify. If $\mu = 0$ (i.e. $M = 0$), then the swinging mass should execute projectile motion: a parabolic path. A parabola is easy to describe in Cartesian coordinates, but the equations of motion are in polar coordinates. So the quickest way to check the projectile solution is first to transform the angular and radial equations to Cartesian coordinates; second, check that a parabola satisfies the transformed equations. The recommended Cartesian transformation places the positive y-axis downward.

$x = r\sin\theta$

$y = r\cos\theta$

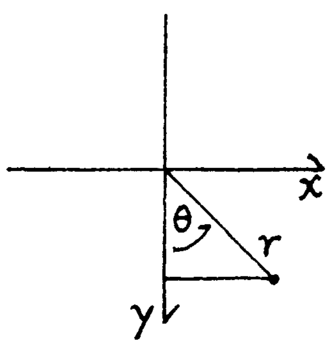


Fig. 2-2

This changes the angular equation (2-10) to

$$\frac{d}{dt}[y\dot{x} - x\dot{y}] = -gx, \quad \text{or}$$

$$x\ddot{y} - y\ddot{x} = gx, \tag{2-12}$$

while the radial equation becomes

$$\mu\frac{(x\dot{y} - y\dot{x})^2}{(x^2 + y^2)} + [1 + \mu](x\ddot{x} + y\ddot{y}) = g[y - \mu(x^2 + y^2)^{\frac{1}{2}}]. \tag{2-13}$$

It is now straight-forward to verify the projectile solution:

ii. <u>Projectile</u> solution

For $\mu = 0$,

$$x(t) = x_0 + \dot{x}_0 t$$

$$y(t) = y_0 + \dot{y}_0 t + \frac{1}{2}gt^2.$$

The <u>peanut</u> solution was found by R. Crandall:

iii. <u>Peanut</u> solution

For $\mu = -4$,

$$\theta(t) = \omega t, \quad \omega = \dot{\theta}_0$$

$$r(t) = \frac{g}{\omega^2}(-4 + \frac{1}{2}\cos\omega t).$$

The peanut is periodic, but unstable in the sense that trajectories near the peanut orbit either spiral into the center or run away. In view of the negative mass, its physical relevance is dubious. No other exact solutions have been discovered so far.

CHAPTER III. The Motions

"id quod visum placet."

> "I never guess", Holmes corrected smoothly. "It is an appalling habit, destructive to the logical faculty."
>
> The Seven-per-cent Solution

I. Numerical Solution

Study of SAM's dynamics begins with extensive numerical solutions of the angular and radial equations. Details about how the computer solves these equations are to be found in Appendix 1. The trajectories drawn by the computer are delightful and often suggest fresh avenues of exploration. Numerical integration reveals many types of motion, from patterns that look random to a wide spectrum of periodic trajectories. Later, numerical solutions serve as a precise yardstick to judge the accuracy of different approximation schemes. Surprisingly, these computer checks show that the numerical results are almost identical to the perturbative solutions (Chapters IV and V) within their domain of application.

I.A. Increasing M

Three typical studies are presented in this section. Each picture frame shows the trajectory of the swinging mass after it starts from rest on the left hand side. The origin is marked by a small cross, usually top of center in each frame. In the first set of pictures, the masses are initially equal. The successive frames show what happens as the nonswinging mass is increased. The increment is arbitrarily chosen to be around one-tenth, from one frame to the next. The initial angle is about 90°, so the swinging bob starts out at the same level as the pulley. The specific initial conditions are:

$m = 1 \qquad M = \mu$

$r_0 = 10 \qquad \theta_0 = 1.57$ radians

$\dot{r}_0 = 0 \qquad \dot{\theta}_0 = 0$

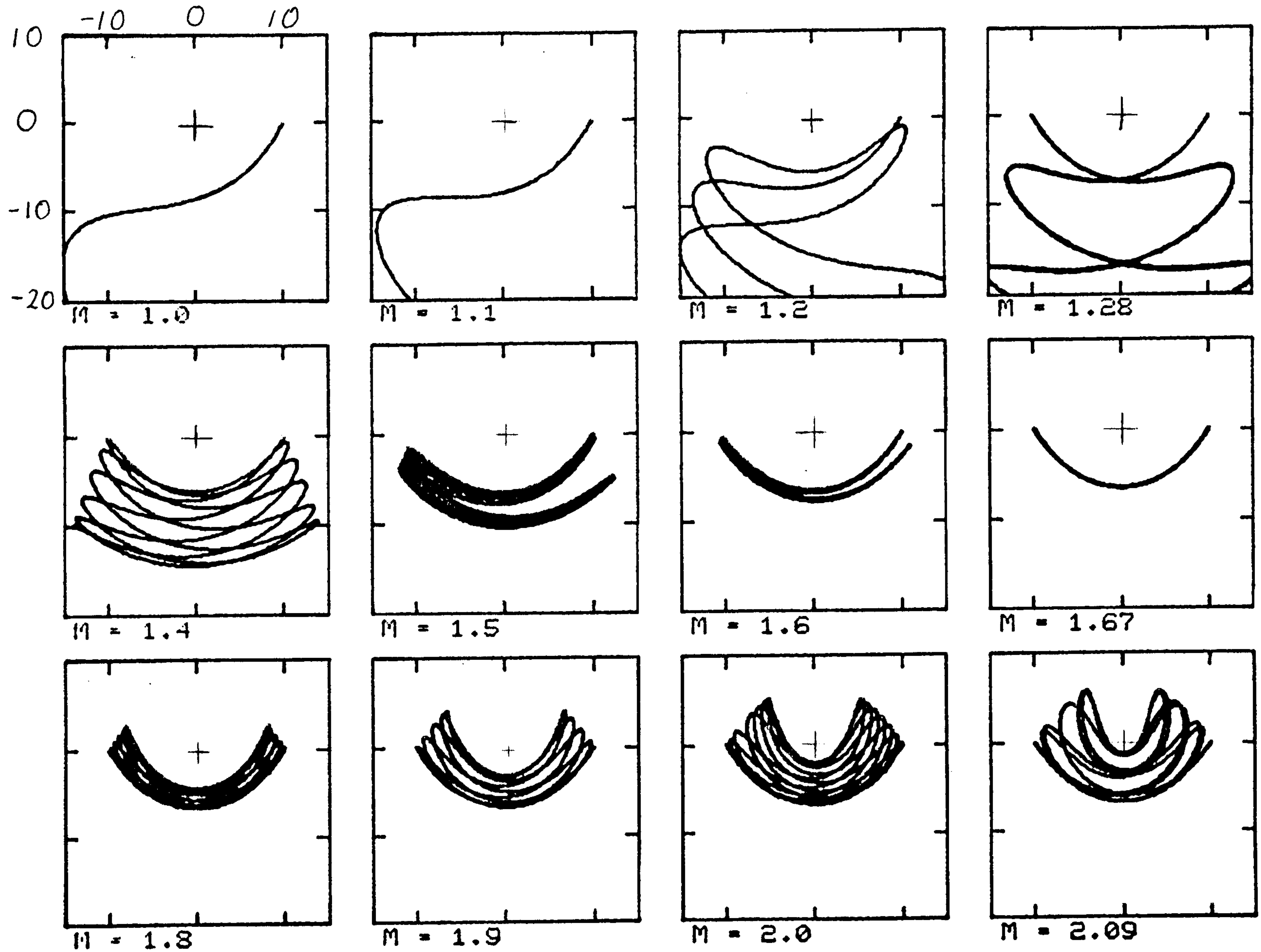
$g = 10$

t: 0 to 10π

M ranges from one to ten and is indicated at the bottom of each frame. The ticks on the boundary of each frame outline a Cartesian grid. The range of the grid is

x-axis: −15 to 15, y-axis: −20 to 10.

The period is 10π in all cases, so if the trajectories look simple it is because the orbit is retracing its path; the motion is periodic. For the first few frames, the entire orbit is not contained completely within the grid.


-10
0
10
10
0
-10
-20
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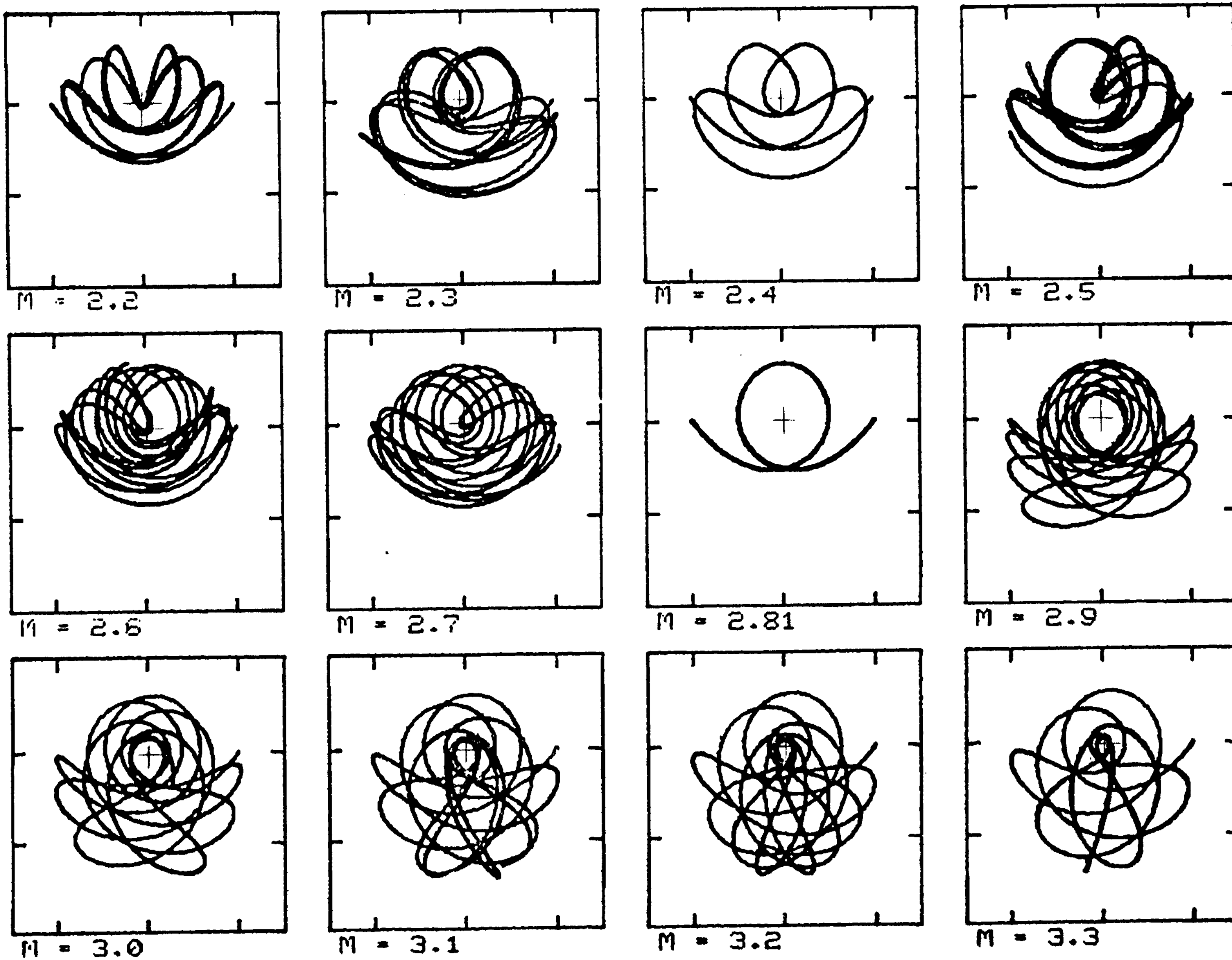
M = 1.1
M = 1.2
M = 1.28
M = 1.4
M = 1.5
M = 1.6
M = 1.67
M = 1.8
M = 1.9
M = 2.0
M = 2.09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M = 2.4
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M = 2.7
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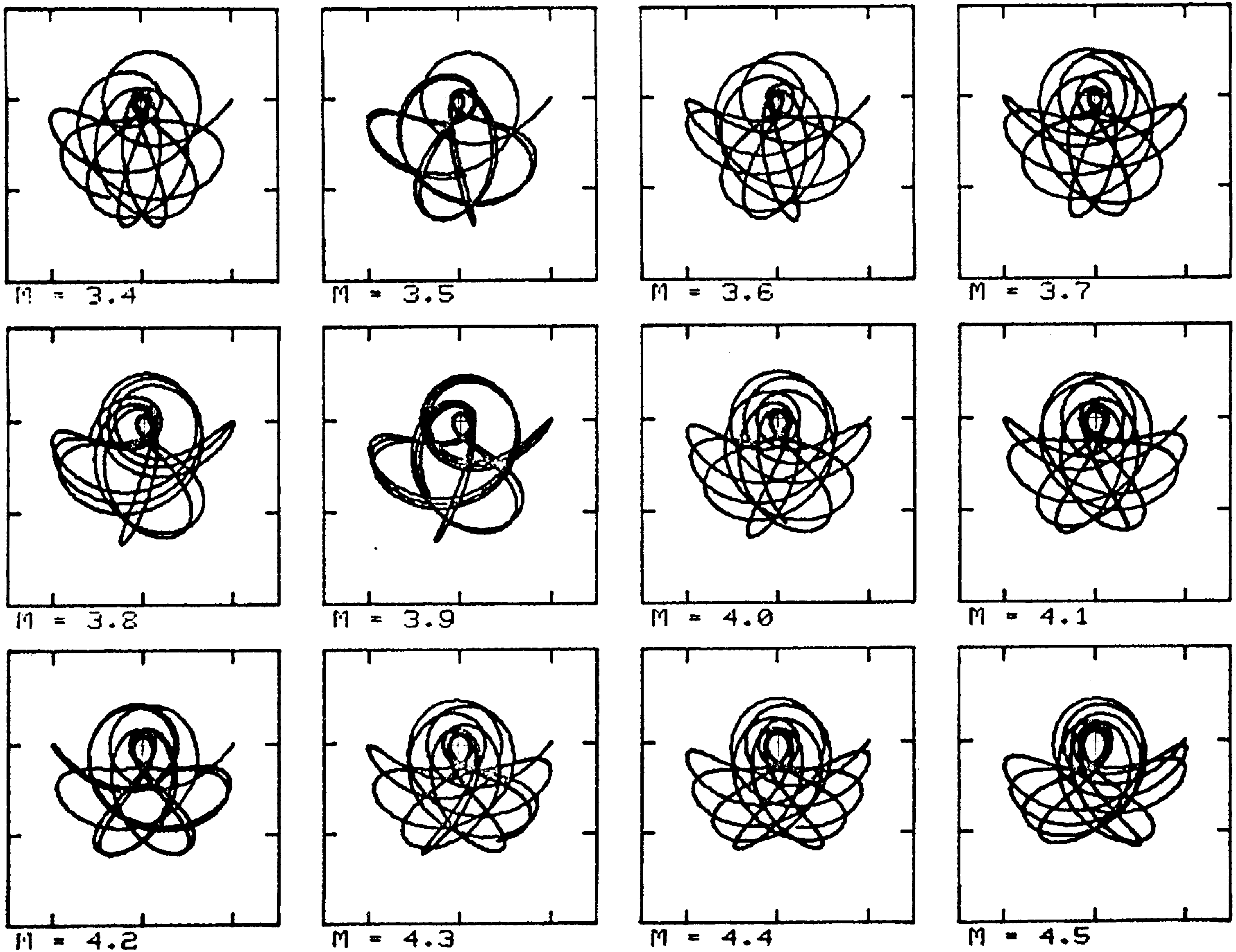
M = 3.1
M = 3.2
M = 3.3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M = 3.7
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M = 4.1
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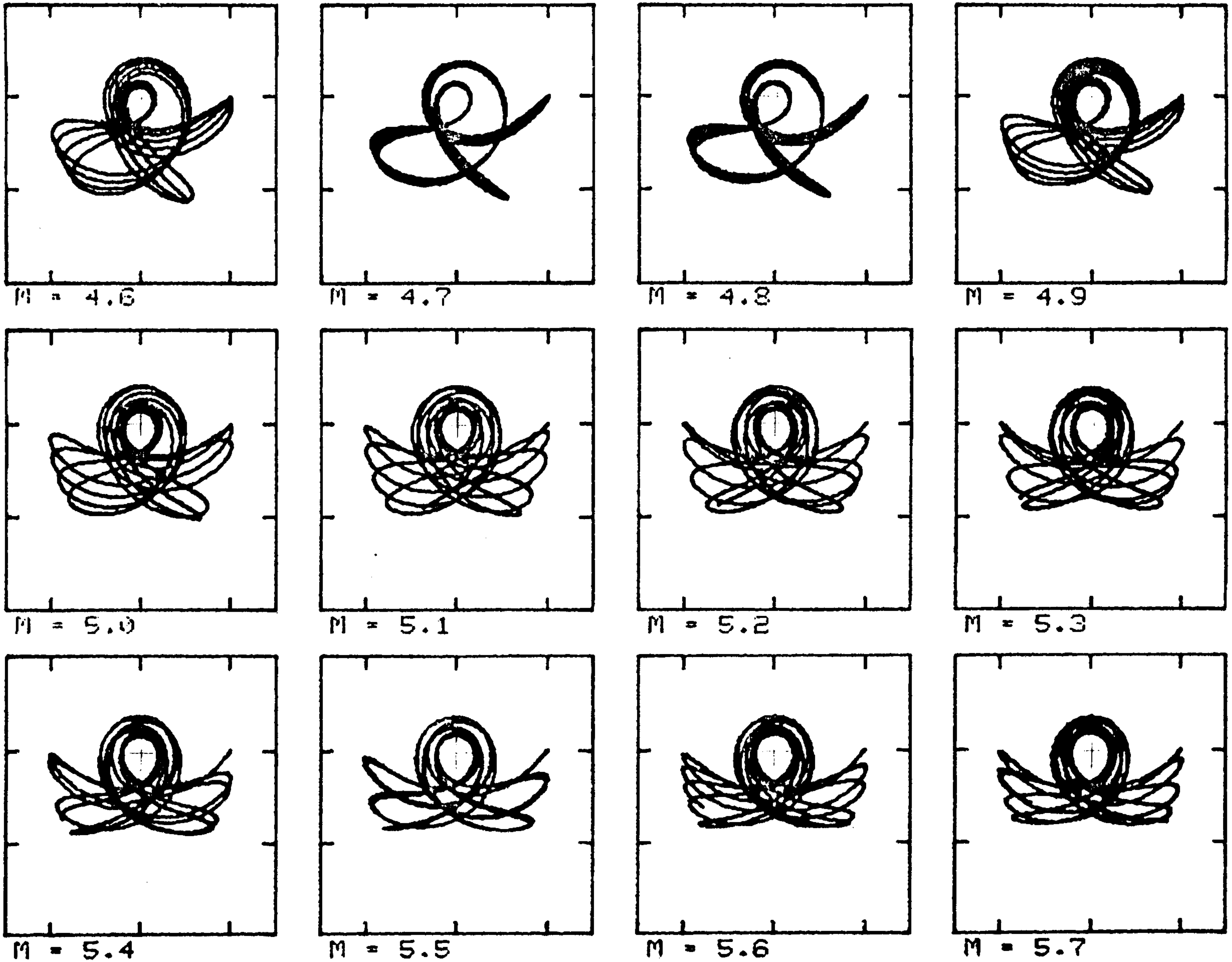
M = 4.3
M = 4.4
M = 4.5



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M = 4.8
M = 4.9
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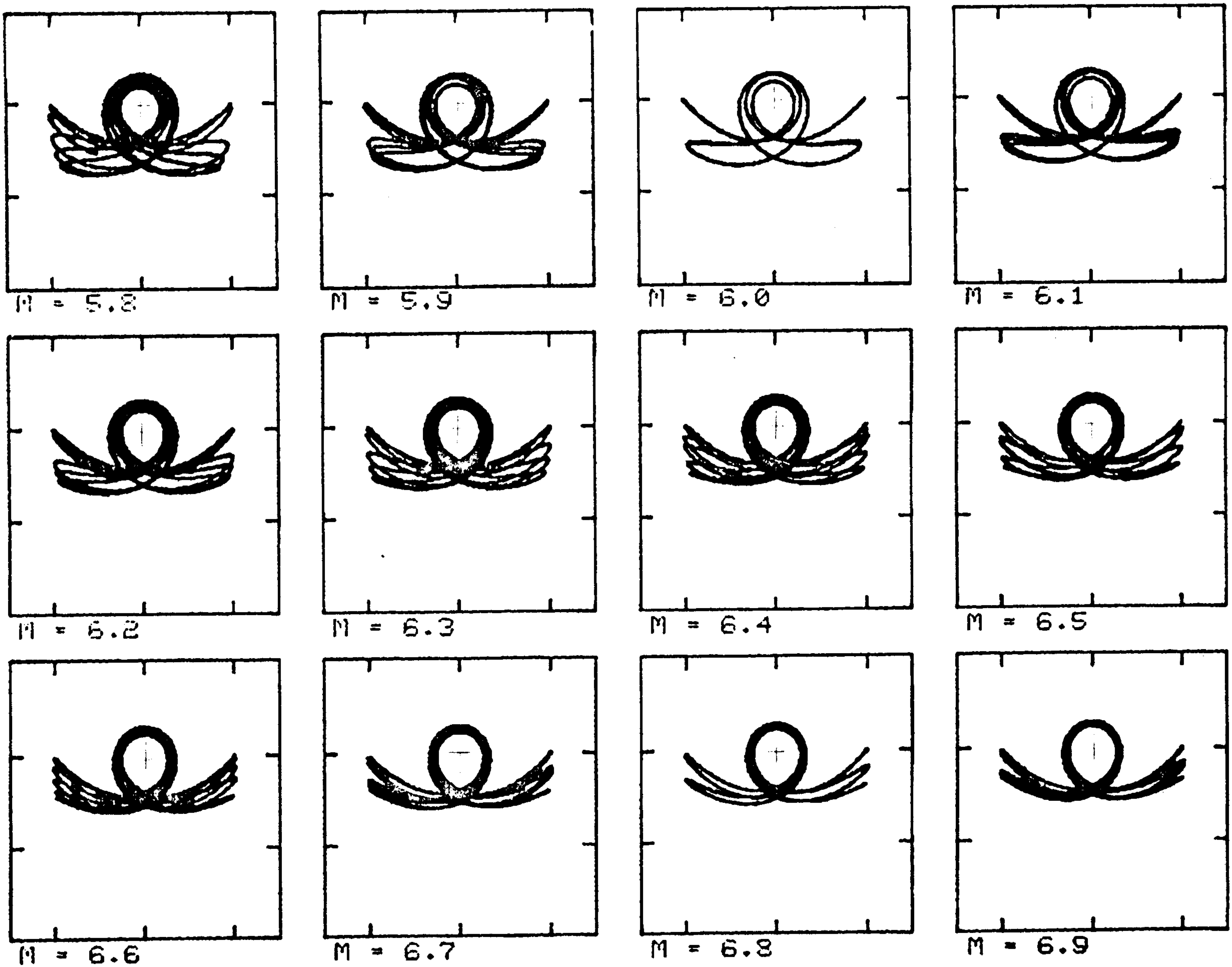
M = 5.6
M = 5.7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M = 6.8
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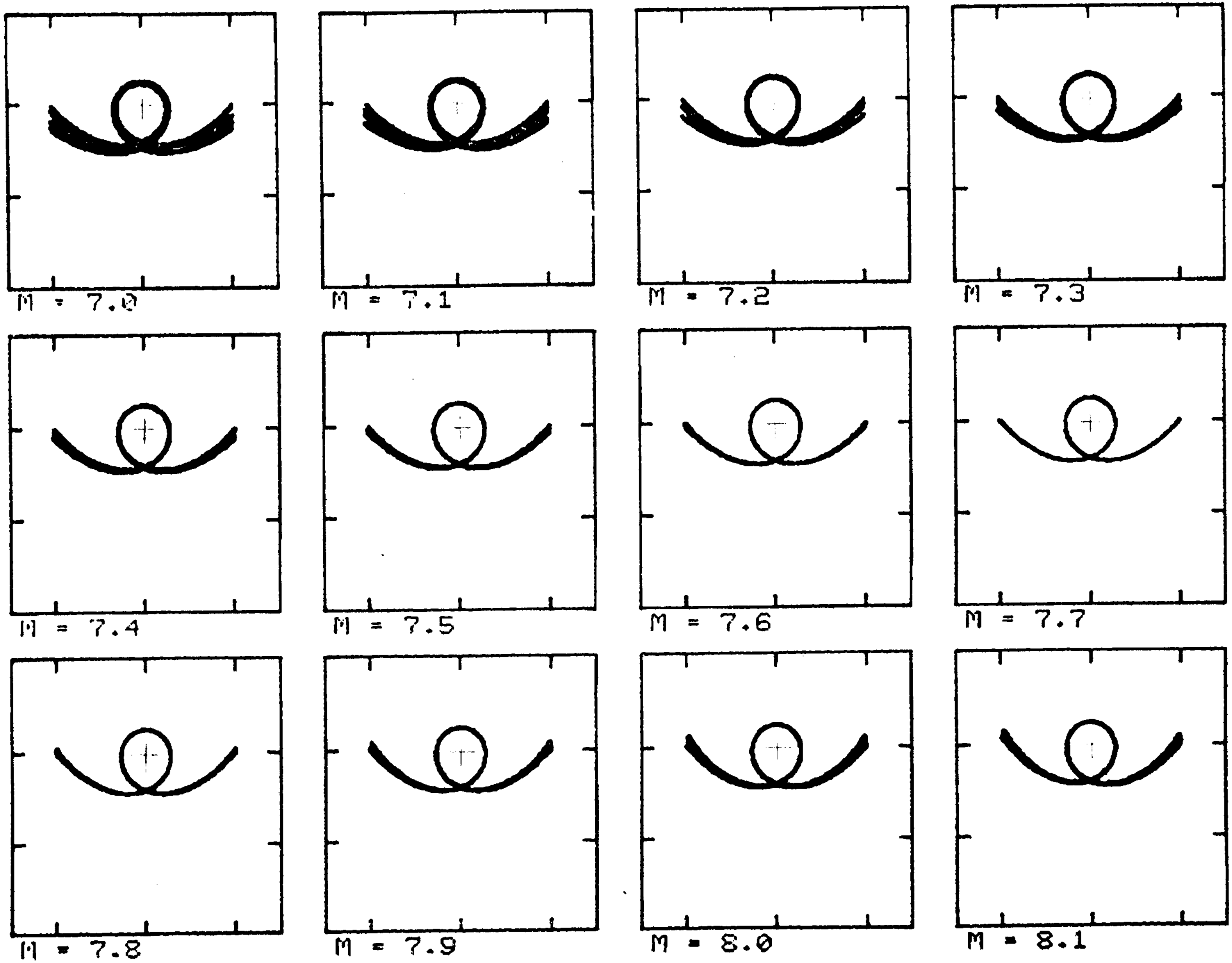
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M = 7.2
M = 7.3
M = 7.4
M = 7.5
M = 7.6
M = 7.7
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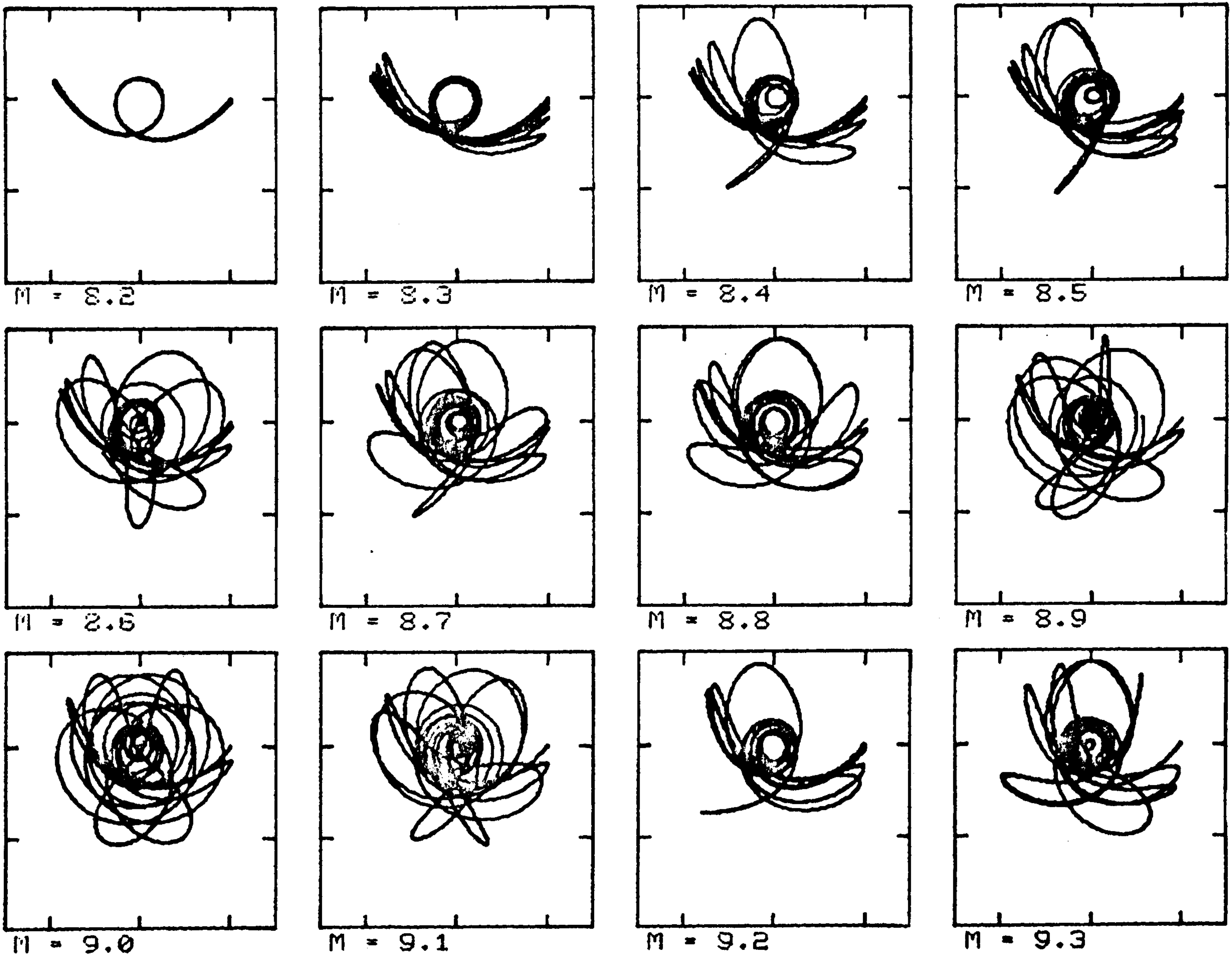
M = 8.0
M = 8.1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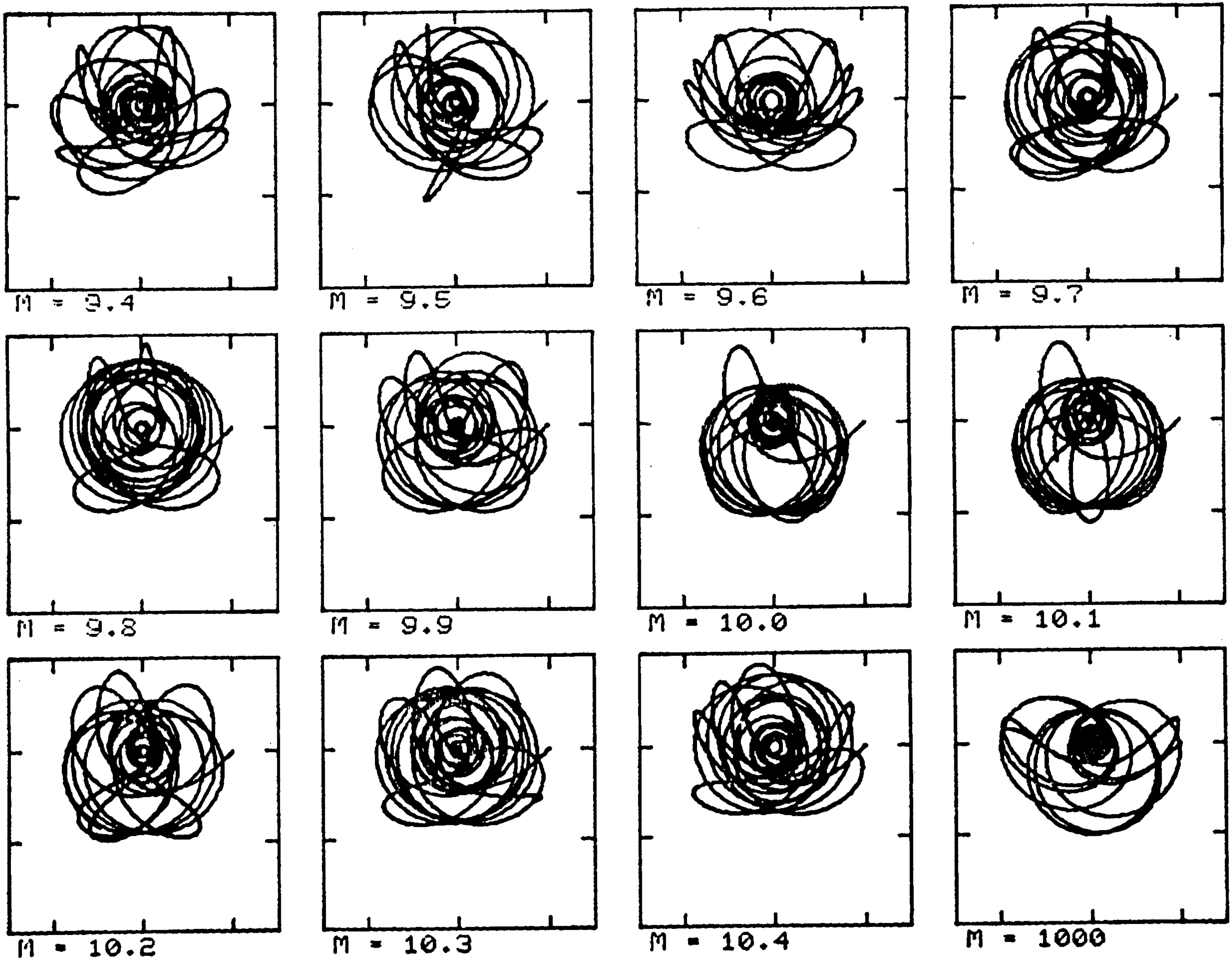
M = 9.3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M = 9.5
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M = 9.7
M = 9.8
M = 9.9
M = 10.0
M = 10.1
M = 10.2
M = 10.3
M = 10.4
M = 1000

Some of the patterns are quite beautiful, and invite special names:

M	Name
1.67	Smile
2.4	Sombrero
2.81	Big Loop
4.7	Whirling Dervish
7.7	Little Loop

The Smile is the simplest instance of a periodic path. The trajectories just named, along with a few others, are all thought to be periodic. Notice that the mass comes dangerously close to the center for M = 2.2 and others.

I.B. Periodic Trajectories

Periodic trajectories are particularly intriguing. This section emphasizes periodic solutions. As in the previous study, M starts at 1 and is increased by small amounts. As M is increased, some of the trajectories retrace their steps, so the path looks periodic. By twiddling M, and examining the new trajectory, it is possible to home in on periodic solutions; i.e., trajectories that look identical over many oscillations. Many trajectories on the next two pages are presumably periodic. The initial conditions for these pictures are:

$m = 1 \qquad M = \mu$

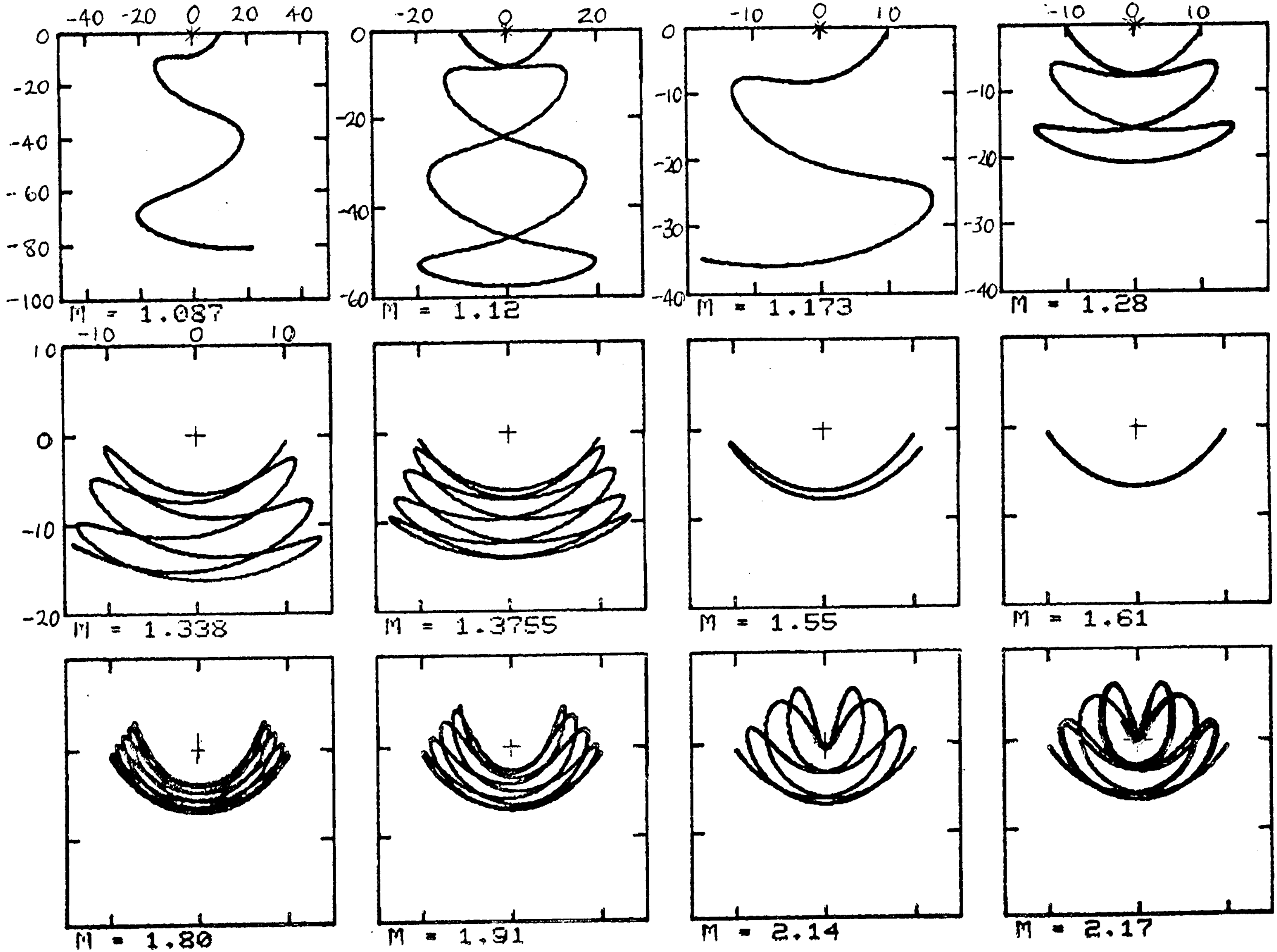
$r_0 = 10 \qquad \theta_0 = 1.5$ radians

$\dot{r}_0 = 0 \qquad \dot{\theta}_0 = 0$

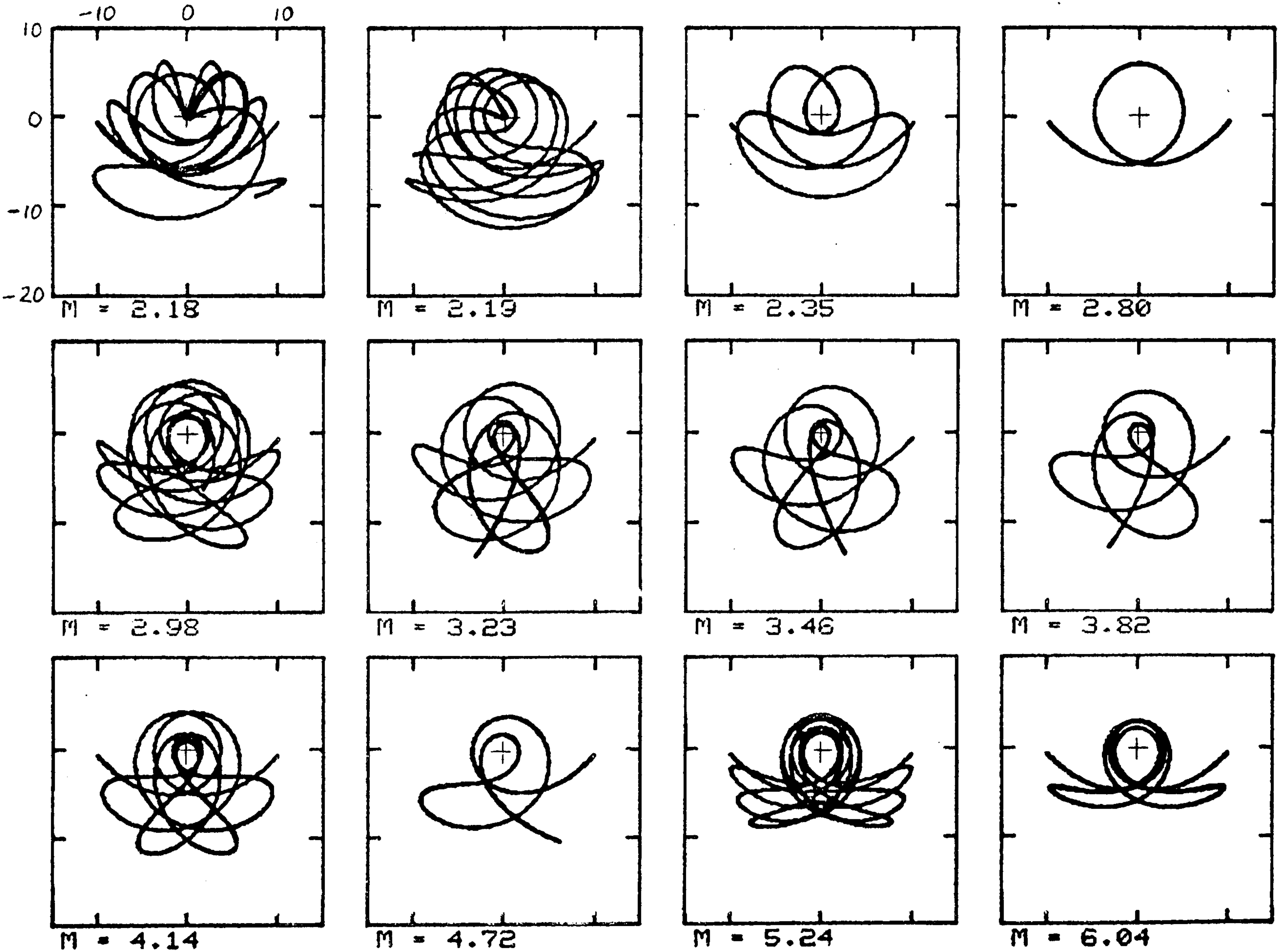
$g = 10$

t: 0 to 5π

Unless indicated otherwise, the grid range is the same as before.



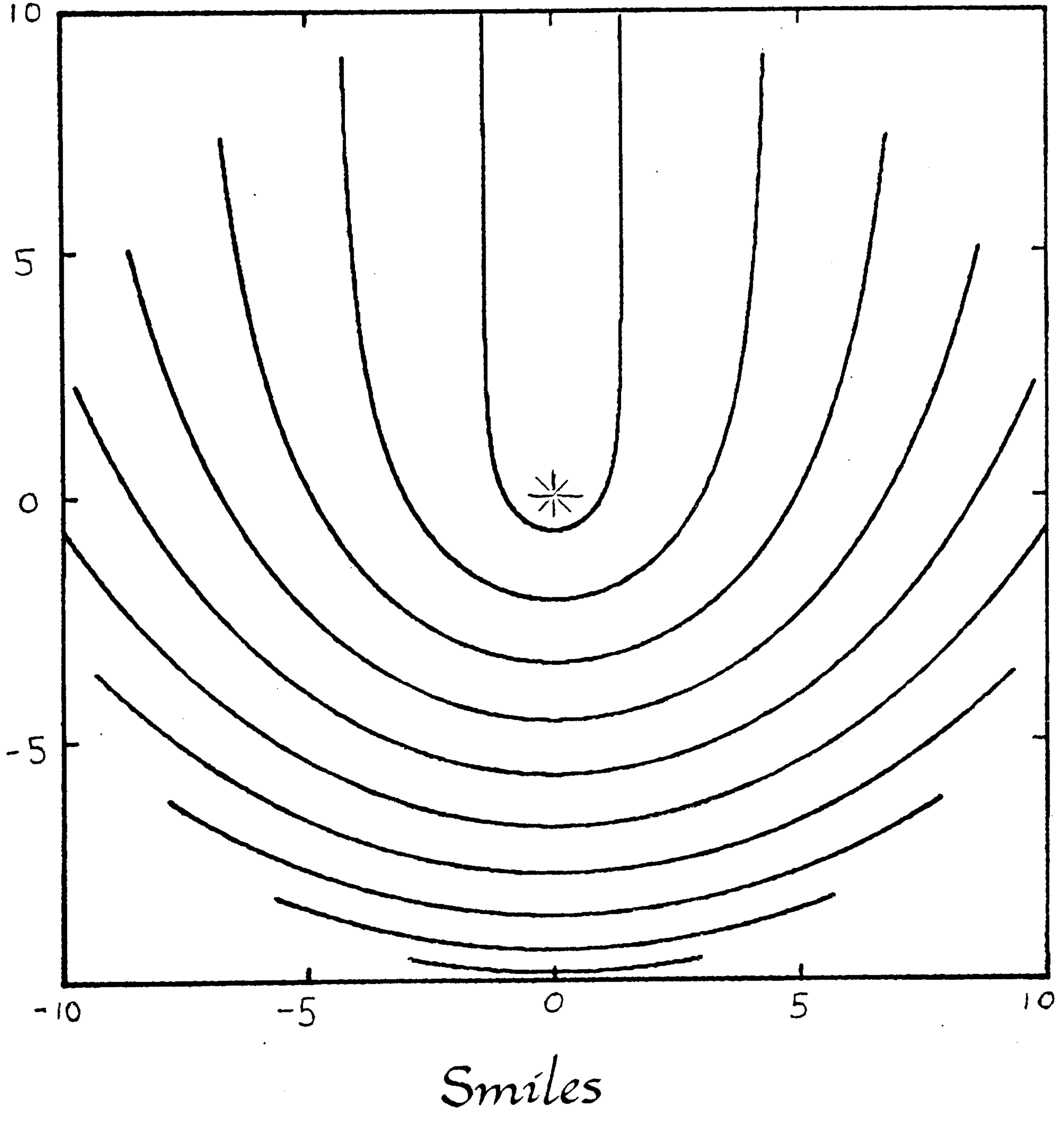
M = 1.087
M = 1.12
M = 1.173
M = 1.28
M = 1.338
M = 1.3755
M = 1.55
M = 1.61
M = 1.80
M = 1.91
M = 2.14
M = 2.17


-10
0
10
10
0
-10
-20
M = 2.18
M = 2.19
M = 2.35
M = 2.80
M = 2.98
M = 3.23
M = 3.46
M = 3.82
M = 4.14
M = 4.72
M = 5.24
M = 6.04

Although this study certainly does not locate all the periodic trajectories within the range of masses tested, a curious pattern is unfolding. Between M = 1 (equal masses) and M = 1.61 (Smile) there appears to be a whole spectrum of periodic solutions, perhaps infinite in number, since the mass difference between two periodic paths decreases as M approaches one. These solutions are alternately symmetric and asymmetric about the vertical axis. For instance, M = 1.12 is symmetric, M = 1.173 is asymmetric; but both look periodic.

I.C. Smiles

The third numerical study focuses on Smiles. The last picture shows smiles for different starting angles. The initial angle, and the M required to obtain a Smile, are indicated on the right. Otherwise, the initial conditions are the same as in the previous study. In particular, r_0 = 10 for all of them.



θ_0 radians	M
3.0	2.97
2.7	2.68
2.4	2.42
2.1	2.15
1.8	1.87
1.5	1.61
1.2	1.38
0.9	1.21
0.6	1.09
0.3	1.02

Initial angle for curves from top to bottom.

II. Periodicity Condition

Smiles exhibit features which are common to other periodic trajectories that start from rest. There exist similar families of Loops, Whirling Dervishes, and other periodic paths. Each curve within the family is characterized by its initial angular displacement, θ_0. For a given θ_0, there is a particular mass ratio μ which leads to a periodic orbit of the specified family:

$$\mu = f(\theta_0); \tag{3-1}$$

this is called the <u>periodicity formula</u> for the family in question. This formula can be constructed by numerical methods. The periodicity formula must be independent of r_0 and g since by Theorem 2.2 the length and the gravitational constant can be eliminated from the angular and radial equations by a suitable change of scale.

For Smiles, the numerical evidence also promotes the following conjectures:

i. $f(0) = 1$

ii. $f(\theta_0)$ is monotonically increasing on 0 to π

iii. $f(\pi) = 3$

III. General Observations

Some of the more salient conclusions that are suggested by numerical studies are: First, when $\mu < 1$, r is unbounded; the solutions run away, like a simple Atwood's machine. Second, if $\mu = 1$, $\dot{r}$ reaches a terminal velocity and θ approaches zero. Third, if $\mu > 1$, then there exists a maximum r; r is bounded above.

One also finds that solutions which start out <u>near</u> periodic trajectories, remain close to them. Motion near periodic orbits is recurrent, in the sense that given any point along the trajectory, the swinging mass will come back close to that point at some later time.

IV. Terminology

The numerical solutions motivate a simple scheme by which to classify the motions of the swinging bob.

<u>bounded</u>: $0 \leq r(t) \leq r_{max}$

<u>stable</u>: $r_{min} \leq r(t) \leq r_{max}$

<u>periodic</u>: $r(t + \tau) = r(t)$

and, $\theta(t + \tau) = \theta(t)$, for some τ.

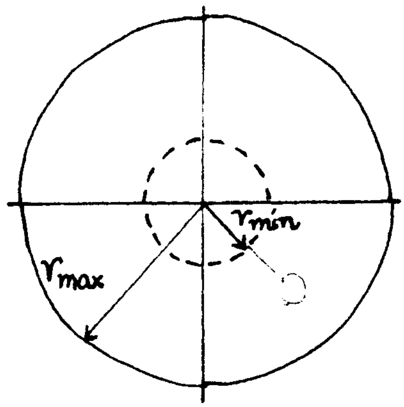


Fig. 3-1

If the bob always remains within some circle of radius $r_{max} < \infty$, the motion is <u>bounded</u>; i.e., there exists an upper bound for r(t). If there further exists an $r_{min} > 0$; (i.e., r has a nonzero lower bound) the motion is said to be <u>stable</u>. Lastly, if r(t) and θ(t) are periodic with commensurable periods then the motion is <u>periodic</u>. Observe that this is a hierarchy: periodic ⇒ stable ⇒ bounded.

V. Bounded Motion

A fundamental theorem about bounded solutions is proved in this section. It says that SAM's motion is bounded if and only if $\mu > 1$; that is, r(t) is bounded only when the swinging mass is lighter than the nonswinging mass.* If $\mu > 1$ the existence of a maximum r follows from

* A trival exception is the simple Atwood's machine that starts

conservation of energy.

THEOREM 3.1

There exists a maximum r when $\mu > 1$.

Proof:

$$E = (1 + \mu)\dot{r}^2 + r^2\dot{\theta}^2 + 2gr(\mu - \cos\theta),$$

$$\geq 2gr(\mu - 1),$$

$$\therefore E \geq 2gr(\mu - 1).$$

This is always true. If $\mu > 1$, it follows that

$$r \leq \frac{E}{2g(\mu - 1)}. \text{ QED} \tag{3-2}$$

In particular, if the system is released from rest, then

$$E = 2gr_0(\mu - \cos\theta_0). \tag{3-3}$$

So an upper bound for r(t) is

$$r_{max} \leq r_0 \frac{\mu - \cos\theta_0}{\mu - 1}. \tag{3-4}$$

A few lemmas are needed to prove the converse. Use brackets to denote the time average of a quantity:

$$\langle x(t)\rangle = \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau x(t)dt. \tag{3-5}$$

Next define

$$T = \mu(g + \ddot{r}), \tag{3-6}$$

---

from rest with equal masses. This is the only equilibrium point of the system.

so that T is proportional to the tension in the string.

If the motion is bounded, then the average velocity and acceleration of M are both zero. This observation is needed for the next two lemmas.

LEMMA 3.1

If the motion is bounded, then $\langle T \rangle = g\mu$

Proof:

$$\langle T \rangle = \langle \mu(g + \ddot{r}) \rangle = \mu g + \mu \langle \ddot{r} \rangle = g\mu. \text{ QED}$$

LEMMA 3.2

If the motion is bounded, then $\langle T\cos\theta \rangle = g$

Proof: Let $y = r\cos\theta$, then

$$\ddot{y} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta.$$

From the angular equation

$$r\ddot{\theta}\sin\theta = -2\dot{r}\dot{\theta}\sin\theta - g\sin^2\theta.$$

Plugging this into the $\ddot{y}$ equation yields

$$\ddot{y} = \ddot{r}\cos\theta - r\dot{\theta}^2\cos\theta + g\sin^2\theta,$$

whereupon, using the radial equation:

$$\ddot{y} = -T\cos\theta + g.$$

For bounded solutions $\langle \ddot{y} \rangle = 0$ so

$$\langle T\cos\theta \rangle = g. \text{ QED}$$

THEOREM 3.2

If r(t) is bounded, then $\mu \geq 1$.

Proof:

$$g = \langle T\cos\theta \rangle \leq \langle T \rangle = g\mu \quad \text{so}$$

$$1 \leq \mu. \text{ QED}$$

VI. Unbounded Motion

The remainder of this chapter will deal with unbounded solutions. The next two chapters are devoted to periodic motion and related behavior.

When $\mu < 1$, the computer suggests that the qualitative behavior of the angular coordinate is similar to that of a damped harmonic oscillator. Insight into the general behavior for $\mu < 1$ is provided by the following approximate calculation.

Consider the case of very small θ, so the equations of motion can be expanded keeping only first-order terms in θ. For motion that starts from rest, the radial equation simplifies to

$$(1 + \mu)\ddot{r} = (1 - \mu)g \implies r(t) = r_0 + \frac{1}{2}at^2 \qquad (3\text{-}7)$$

where a is the Atwood acceleration

$$a = \frac{1 - \mu}{1 + \mu}\, g.$$

The angular equation is

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\theta = 0.$$

Next consider a large t approximation $t \gg 0$, i.e. look at the motion a long time after the bob is released. Assuming $\frac{1}{2}at^2 \gg r_0$, so the length the length of the pendulum has changed much compared to its original value, we have:

$$\ddot{\theta} + \frac{4}{t}\dot{\theta} + \frac{2}{t^2}\frac{1 + \mu}{1 - \mu}\theta = 0.$$

The reader can verify that an exact solution to this equation is

$$\theta = \frac{e^{\alpha \ln(t)}}{t^2}, \quad \alpha = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{4\mu}{1 - \mu}}. \tag{3-8}$$

If $\mu \leq \frac{1}{17}$, then α is real and θ crosses the vertical axis at ∞. Otherwise, when $\frac{1}{17} < \mu < 1$, α is imaginary. Since this differential equation is linear, both the real and imaginary parts are solutions. Using Euler's celebrated identity -- $e^{i\theta} = \cos\theta + i\sin\theta$ -- gives $\theta(t) = \frac{\cos(\alpha \ln t)}{t^2}$.

VII. Terminal Velocity

For $\mu = 1$ and motion starting from rest, both the numerical evidence and physical intuition suggest that $\dot{r}(t)$ reaches a terminal velocity as $\theta(t)$ approaches zero. This terminal velocity can be derived by energy conservation. Consider the initial and final configurations for equal masses:

Equal Masses

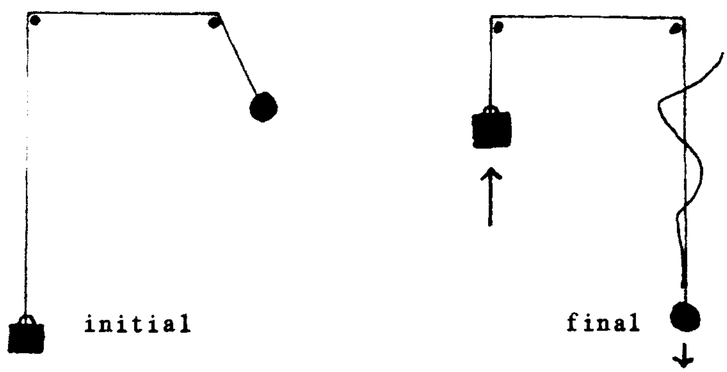


Fig. 3-2

Looking at Fig. 3-2, one sees that the potential energy it takes to lift the swinging mass away from the vertical ends up in the form of kinetic energy. The initial potential energy is

$$E_i = mgr_0(1 - \cos\theta_0).$$

while the final kinetic energy is

$$E_f = m\dot{r}^2.$$

Setting $E_i = E_f$ and solving for the velocity gives

$$\dot{r}_{terminal} = \sqrt{gr_0(1 - \cos\theta_0)}. \tag{3-9}$$

CHAPTER IV. Smiles

"And then their features started into smiles,
sweet as blue heavens o'er enchanted isles."

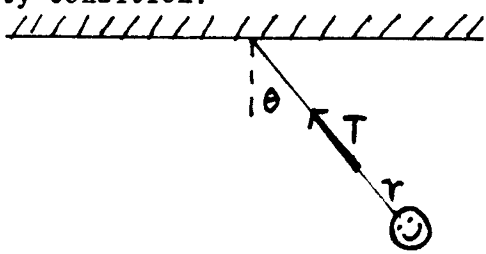
John Keats

Smile, damn you; smile!

American saying, c. 1910

I. Periodicity Formula for Smiles

It is easy to discover a small-angle periodicity formula for Smiles by studying the average tension in a simple pendulum. For small angles, a Smiles' trajectory is almost that of a simple pendulum, since the string length changes only slightly over one period. Now, the (time) average of the tension in the string equals Mg for any bounded motion. Setting the average tension in the simple pendulum equal to Mg leads to the desired periodicity condition.



Simple Pendulum

Fig. 4-1

The tension in a simple pendulum is

$$T = mr\dot{\theta}^2 + mg\cos\theta.$$

For small angles, the simple pendulum executes simple harmonic motion:

$$\theta = \theta_0 \cos(\omega t), \quad \omega^2 = \frac{g}{r}.$$

The average tension is therefore:

$$\langle T\rangle = mr\theta_0^2\omega^2\langle\sin^2\omega t\rangle + mg\langle\cos(\theta_0\cos\omega t)\rangle$$
$$= mg\{\theta_0^2\langle\sin^2\omega t\rangle + \langle\cos(\theta_0\cos\omega t)\rangle\}.$$

The argument of $\cos(\theta_0\cos\omega t)$ is small; keeping terms of order θ_0^2 in the Taylor expansion gives

$$\langle T\rangle = mg\{\theta_0^2\langle\sin^2\omega t\rangle + 1 - \frac{\theta_0^2}{2}\langle\cos^2\omega t\rangle\}$$
$$= mg(1 + \frac{\theta_0^2}{4}),$$

since $\langle\sin^2\omega t\rangle = \langle\cos^2\omega t\rangle = \frac{1}{2}$. Setting $\langle T\rangle = Mg$ yields the periodicity formula

$$\mu = 1 + \frac{1}{4}\theta_0^2. \qquad (4\text{-}1)$$

The small-angle approximation was used to calculate the integral of $\cos(\theta_0\cos\omega t)$. In fact, this is not necessary since the solution is exactly a Bessel function of order 0:

$$\langle\cos(\theta_0\cos\omega t)\rangle = \frac{1}{T}\int_0^T\cos(\theta_0\cos\omega t)\,dt = J_0(\theta_0).$$

Using this Bessel function leads to the periodicity formula,

$$\mu = \frac{\theta_0^2}{2} + J_0(\theta_0) \qquad (4\text{-}2)$$

which is consistent with (4-1) for small angles.*

The periodicity formula can be constructed to an arbitrary degree of accuracy by the computer. The results are presented in the following

* Keeping only the first two terms in the Taylor expansion,

$$J_0(\theta_0) = 1 - \frac{1}{4}\theta_0^2 + \frac{1}{64}\theta_0^4 - \ldots .$$

table; this is the data used in section I.C, Chapter III.

Smile Solutions

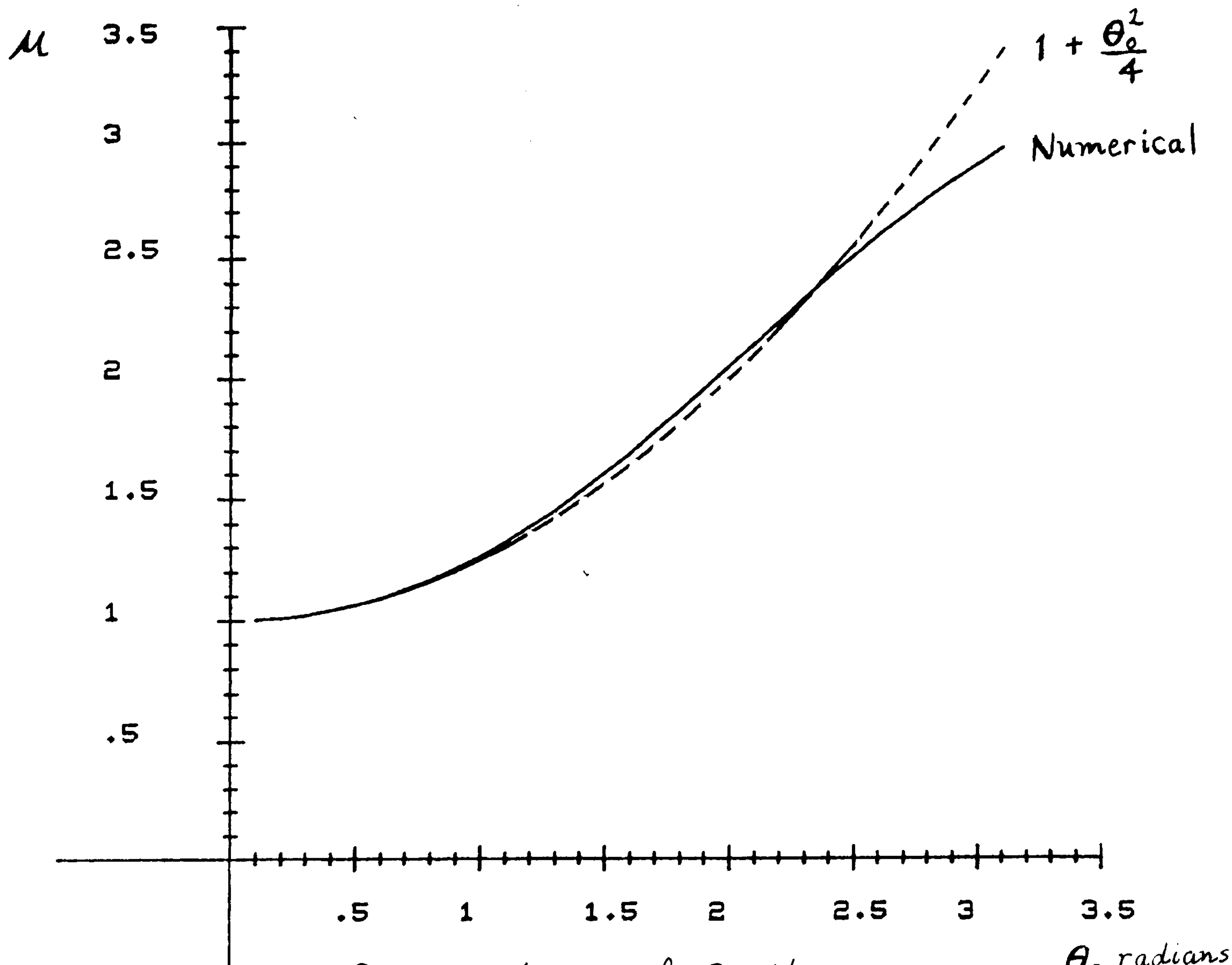
Angle	Mass ratio	Minimum r	Maximum r	Angular Frequency
0.1	1.002502	9.981274	10.000000	1.000668
0.2	1.010027	9.925389	10.000000	1.002373
0.3	1.022637	9.833211	10.000000	1.004819
0.4	1.040427	9.706154	10.000000	1.008510
0.5	1.063523	9.546141	9.999996	1.013225
0.6	1.092068	9.355539	9.999998	1.018740
0.7	1.126214	9.137073	10.000002	1.024953
0.8	1.166103	8.893732	10.000000	1.031759
0.9	1.211845	8.628649	9.999999	1.039179
1.0	1.263503	8.344969	10.000000	1.046841
1.1	1.321066	8.045718	9.999999	1.054616
1.2	1.384436	7.733667	9.999997	1.062439
1.3	1.453404	7.411225	9.999999	1.070101
1.4	1.527647	7.080346	9.999999	1.077381
1.5	1.606727	6.742481	10.000000	1.084046
1.6	1.690092	6.398557	10.000000	1.090217
1.7	1.777096	6.048997	9.999998	1.095729
1.8	1.867017	5.693772	10.000000	1.100487
1.9	1.959080	5.332466	9.999998	1.104398
2.0	2.052481	4.964356	10.000000	1.107517
2.1	2.146415	4.588506	9.999999	1.109980
2.2	2.240097	4.203844	9.999999	1.111647
2.3	2.332783	3.809247	9.999998	1.112717
2.4	2.423785	3.403621	10.000000	1.113281
2.5	2.512484	2.985976	9.999998	1.113394
2.6	2.598347	2.555506	10.000000	1.113149
2.7	2.680929	2.111669	10.000000	1.112717
2.8	2.759885	1.654286	9.999999	1.112116
2.9	2.834975	1.183666	10.000000	1.111516
3.0	2.906076	0.700766	10.000000	1.111028
3.1	2.973206	0.207481	10.000000	1.110750

The periodicity condition is read from the angle and mass ratio columns. The minimal value of r occurs at $\theta(t) = 0$, and the maximum value at $\theta(t) = \theta_0$. The computer decides if a given path is a Smile by checking to see that the maximum values of the radial coordinate are identical over many oscillations. The degree to which the computer maintains constant amplitude motion is indicated by the slight variations in the maximum r column. Evidently, the solutions are good to six significant digits.

A comparison of the Smile periodicity formulas is shown in the next table. As expected there is good agreement for small angles, $\theta_0 < 0.5$ radians. More surprisingly, the theoretical formulas remain quite good even for relatively large angles. Notice that the Bessel function formula is distinctly better, up to angles of 1.5 radians; above this, neither formula is terribly good, though (4-1) is somewhat better.

Comparison of Smile Periodicity Formulas

Angle	Numerical	$1 + \frac{\theta_0^2}{4}$	$J_0(\theta_0) + \frac{\theta_0^2}{2}$
0.1	1.002502	1.002500	1.002502
0.2	1.010027	1.010000	1.010025
0.3	1.022637	1.022500	1.022626
0.4	1.040427	1.040000	1.040398
0.5	1.063523	1.062500	1.063470
0.6	1.092068	1.090000	1.092005
0.7	1.126214	1.122500	1.126201
0.8	1.166103	1.160000	1.166287
0.9	1.211845	1.202500	1.212524
1.0	1.263503	1.250000	1.265198
1.1	1.321066	1.302500	1.324622
1.2	1.384436	1.360000	1.391133
1.3	1.453404	1.422500	1.465086
1.4	1.527647	1.490000	1.546855
1.5	1.606727	1.562500	1.636828
1.6	1.690092	1.640000	1.735402
1.7	1.777096	1.722500	1.842985
1.8	1.867017	1.810000	1.959986
1.9	1.959080	1.902500	2.086819
2.0	2.052481	2.000000	2.223891
2.1	2.146415	2.102500	2.371607
2.2	2.240097	2.210000	2.530362
2.3	2.332783	2.322500	2.700540
2.4	2.423785	2.440000	2.882508
2.5	2.512484	2.562500	3.076616
2.6	2.598347	2.690000	3.283195
2.7	2.680929	2.822500	3.502551
2.8	2.759885	2.960000	3.734964
2.9	2.834975	3.102500	3.980688
3.0	2.906076	3.250000	4.239948
3.1	2.973206	3.402500	4.512936



Comparison of Smile Periodicity Formulas

II. Almost Simple Pendulum

The approximate form of the Smile trajectories can be obtained by a perturbation scheme, involving expansion in powers of θ_0. The key idea is to exchange increasingly higher order solutions between the radial and angular equations. A "zeroth-order" solution of the radial equation is plugged into the angular equation so that, in this order, the angular equation can be solved exactly. This solution is plugged back into the radial equation, yielding a "first order" solution. In principle, this boot-strapping technique can be continued ad infinitum, resulting in solutions to any order of accuracy.

A Smile trajectory is close to the path of a pendulum for small angles. So to solve SAM by the perturbation method assume

$$\theta \ll 1 \text{ and } r = r_0\{1 + \varepsilon(t)\} \text{ with } \varepsilon \ll 1. \tag{4-3}$$

The equations of motion are:

$$(1 + \mu)\ddot{r} = r\dot{\theta}^2 + g(\cos\theta - \mu)$$
$$\frac{d}{dt}(r^2\dot{\theta}) = -gr\sin\theta$$

The order of the approximation is defined by the power of θ_0. It will turn out that ε is of order θ_0^2. To <u>first-order</u> (i.e., keeping terms of order θ_0 and less) $r = r_0$. Plugging this into the angular equation gives simple harmonic oscillation as expected.

<u>First-order Smile</u>:

$$r = r_0 \tag{4-4}$$

$$\theta = \theta_0\cos(\omega t), \quad \omega^2 = \frac{g}{r_0} \tag{4-5}$$

Expanding the radial equation to second order in θ gives the

<u>second</u>-<u>order</u> equation:

$$(1 + \mu)\ddot{r} = r\dot{\theta}^2 + g(1 - \frac{\theta^2}{2} - \mu).$$

Applying (4-3) and (4-5) gives

$$(1 + \mu)r_0\ddot{\varepsilon} = r_0(1 + \varepsilon)\theta_0^2\omega^2\sin^2\omega t + g(1 - \mu) - \frac{g}{2}\theta_0^2\cos^2\omega t.$$

The term containing ε on the right is of fourth-order in θ_0 and therefore dropped. Solving for $\ddot{\varepsilon}$ yields

$$\ddot{\varepsilon} = \frac{\omega^2}{(1 + \mu)}[1 - \mu + \frac{\theta_0^2}{4}(1 - 3\cos(2\omega t)]$$

The solution to this differential equation is $\varepsilon = A + Bt + f$, where f is a solution to the inhomogeneous equation:

$$f(t) = Ct^2 + D\cos^2(\omega t).$$

Differentiating f and solving for C and D yields

$$\varepsilon(t) = A + Bt + \frac{\omega^2}{2(1 + \mu)}[1 - \mu + \frac{\theta_0^2}{4}]t^2 + \frac{3}{16}\frac{\theta_0^2}{(1 + \mu)}\cos(2\omega t).$$

A and B are determined by the initial conditions $\varepsilon(0) = \dot{\varepsilon}(0) = 0$, which imply

$$\varepsilon = \frac{\omega^2}{2(1 + \mu)}[1 - \mu + \frac{\theta_0^2}{4}]t^2 - \frac{3}{8}\frac{\theta_0^2}{(1 + \mu)}\sin^2(\omega t).$$

This result holds for short times after release, provided $\theta_0 \ll 1$. After that the t^2 builds up, violating the condition $\varepsilon \ll 1$. The solution holds for all times if the radius never changes much. This means $(1 - \mu) + \frac{\theta_0^2}{4} = 0$ -- i.e. $\mu = 1 + \frac{\theta_0^2}{4}$ -- reproducing the small angle periodicity condition. If $\mu = 1 + \frac{\theta_0^2}{4}$ and $\theta_0 \ll 1$ then

$$r(t) = r_0(1 - \frac{3}{2}\frac{\mu - 1}{\mu + 1}\sin^2\omega t)$$

$$= r_0(1 - \frac{3}{16}\,\theta_0^2 \sin^2\omega t).$$

To finish the program and get a complete Smile, put the first-order radial solution back into the angular equation and solve for $\theta(t)$ in third-order:

$$\frac{d}{dt}[r_0^2(1 + 2\varepsilon + \varepsilon^2)\dot{\theta}] = -gr_0(1 + \varepsilon)[\theta - \frac{\theta^3}{6}].$$

Keeping only the third order terms (θ is of order θ_0, ε is of order θ_0^2) gives

$$\frac{d}{dt}[(1 + 2\varepsilon)\dot{\theta}] = -\omega^2\theta(1 + \varepsilon - \frac{\theta^2}{6}).$$

In the parentheses on the right it suffices to set $\theta = \theta_0 \cos(\omega t)$:

$$1 + \varepsilon - \frac{\theta^2}{6} = 1 - \frac{3}{8}\,\frac{\theta_0^2}{(1 + \mu)}\sin^2\omega t - \frac{1}{6}\,\theta_0^2 \cos^2\omega t$$

$$= 1 - \frac{\theta_0^2}{96}\,\{17 - \cos(2\omega t\}$$

when $1 + \mu$ is replaced by $2 + \frac{\theta_0^2}{4} = 2$, to this order. The next step is to solve

$$\frac{d}{dt}[(1 + 2\varepsilon)\dot{\theta}] = -\omega^2\theta\,[1 - \frac{\theta_0^2}{96}\{17 - \cos(2\omega t)\}].$$

Try a solution of the form

$$\theta(t) = \theta_0\{\cos(\bar{\omega}t) + \delta(t)\}$$

where $\delta(t)$ is of order θ_0 or smaller, and $\bar{\omega} = \omega + \omega\lambda$, $\lambda \ll 1$. A rather long calculation leads to the equation

$$\ddot{\delta} + \omega^2\delta = \frac{1}{192}\,\omega^2\theta_0^2\{-21\cos(\omega t) + 53\cos(3\omega t)\}. - (\omega^2 - \bar{\omega}^2)\cos\bar{\omega}t$$

Thus $\omega^2 - \bar{\omega}^2 = \frac{-7}{64}\,\omega^2\theta_0^2$. Let $\bar{\omega} = \omega(1 + \gamma)$, then $\gamma = \frac{7}{64}\,\theta_0^2$. The conclusion is

$$\bar{\omega} = \omega[1 + \frac{7}{120}\,\theta_0^2],$$

$$\ddot{\delta} + \omega^2\delta = \frac{53}{192}\,\omega^2\theta_0^2\cos(3\omega t).$$

The particular solution to the inhomogeneous equation is

$$\delta = \frac{-53}{1536}\,\theta_0^2\cos(3\omega t).$$

Therefore, the general solution is

$$\delta(t) = A\cos(\omega t) + B\sin(\omega t) - \frac{53}{1536}\,\theta_0^2\cos(3\omega t).$$

Apply the initial conditions $\delta(0) = \dot{\delta}(0) = 0$ to get the desired result. At last, the Smile to order θ_0^3 is

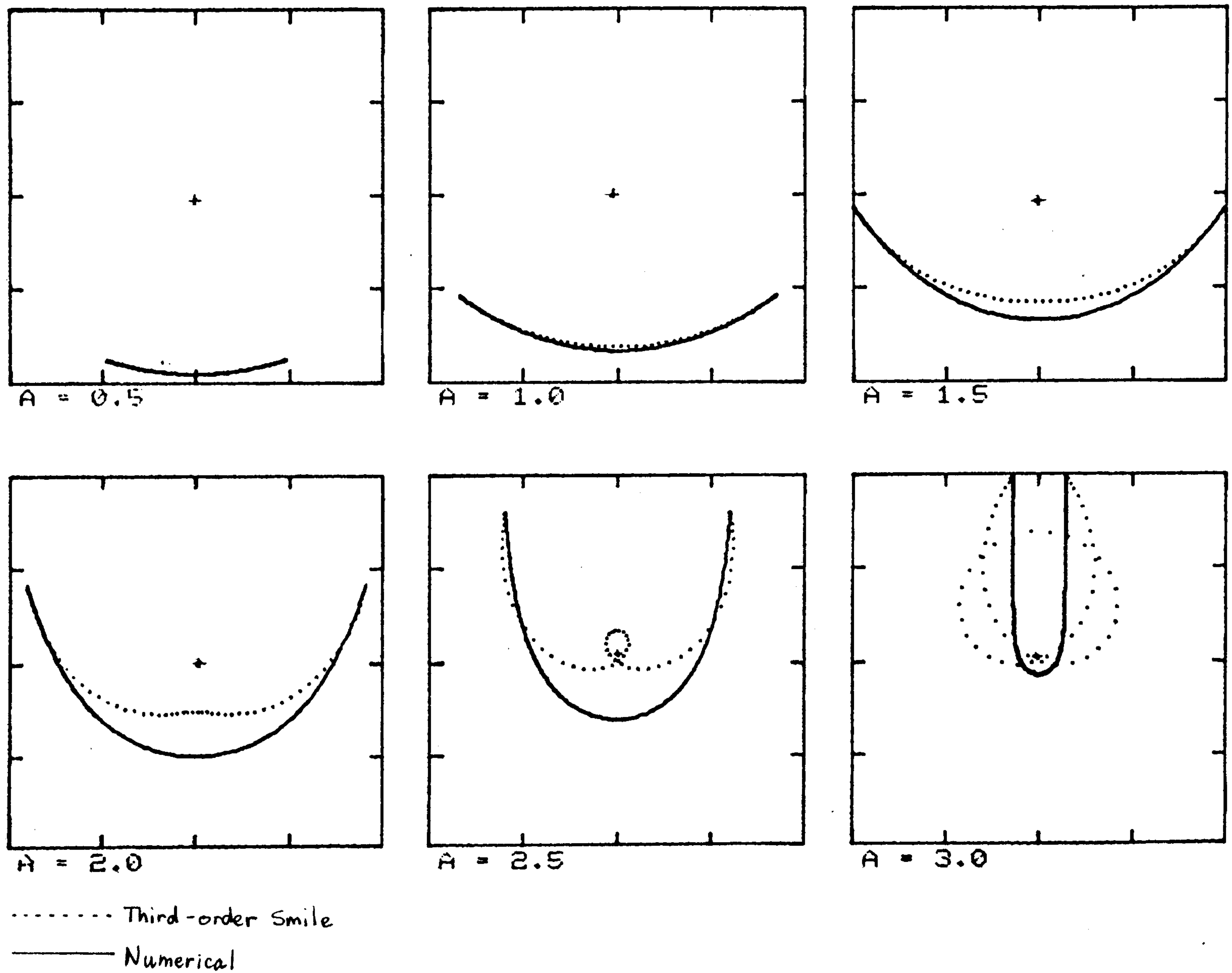
<u>Third-order Smile</u>:

$$r(t) = r_0\,\{1 - \frac{3}{16}\,\theta_0^2\sin^2(\bar{\omega}t)\} \tag{4-6}$$

$$\theta(t) = \theta_0\cos(\bar{\omega}t) + \frac{53}{1536}\,\theta_0^2\,\{\cos(\bar{\omega}t) - \cos(3\bar{\omega}t)\} \tag{4-7}$$

$$\bar{\omega} = \sqrt{\frac{g}{r_0}}(1 + \frac{7}{128}\,\theta_0^2) \tag{4-8}$$

This chapter concludes with a comparison between a Smile given by numerical solution and the perturbation method. The trajectories of both are illustrated on the next page. The dotted line shows the perturbative path -- equations (4-6) through (4-8) -- and the solid line indicates the numerical (true) trajectory. The starting angle in radians is indicated below each frame. The pulley is in the dead center. For $\theta_0 = 0.5$ radians, the solutions look identical. As θ_0 increases, the difference between the two solutions becomes evident. Of course, for $\theta_0 > 1.0$ radians, there is no reason to expect that the perturbative solution will fit the true path; despite this, the agreement is not bad.



Â = 0.5
Â = 1.0
Â = 1.5
Â = 2.0
Â = 2.5
Â = 3.0
Third-order Smile
Numerical

CHAPTER V. Teardrops

"He calleth to me out of Seir,
Watchman, what of the night?
The watchman said, the morning cometh,
and also the night;
if ye enquire, enquire ye;
return, come."

Isaiah

"Hinc illae lacrymae."

Horace

I. Almost Atwood's Solutions

In Chapter IV, Smiles were obtained as perturbations from the simple pendulum. This chapter analyses Teardrops as perturbations on the Atwood's machine. Unlike Smiles, the Atwood's solutions do not start from rest, but are outwardbound from the pulley.

Consider the case in which mass m is fired out radially from the origin, at an angle θ_0. If $\theta_0 = 0$, then the swinging mass is fired straight down, and if $\mu > 1$, it will return to the pulley in a time specified by the Atwood solution. Now, if $\theta_0 << 1$, then one can solve for motion close to an Atwood's machine by the perturbation method illustrated in the Smile solutions.

The solution to the simple Atwood's machine with initial downward velocity v is

$$\theta = 0; \quad \mu > 1.$$

$$(1 + \mu)\ddot{r} = g(1 - \mu) \Rightarrow r = vt - \frac{1}{2}at^2$$

$$a = \frac{\mu - 1}{\mu + 1}g$$

The solution pertains for

$$0 \le t \le \frac{2v}{a};$$

at the latter time it is back at the pulley.

Consider a small deviation from the simple Atwood's machine with $\theta \ll 1$. To first-order in θ, the radial equation is still unchanged. However, the angular equation reads:

$$\frac{d}{dt}(r^2\dot{\theta}) = -gr\sin\theta = -gr\theta.$$

Differentiating yields

$$r\ddot{\theta} + 2\dot{r}\dot{\theta} = -g\theta.$$

Plugging in the Atwood solution results in a first-order angular equation

$$(vt - \frac{1}{2}at^2)\ddot{\theta} + 2(v - at)\dot{\theta} + g\theta = 0 \qquad (5\text{-}1)$$

The next step is to try a power-series solution to (5-1).

Let $\theta = \sum_{n=0}^{\infty} b_n t^n$, then

$$(vt-\frac{1}{2}at^2)\sum_{n=1}^{\infty} b_n n(n-1)\, t^{n-2} + 2(v-at)\sum_{n=0}^{\infty} b_n n\, t^{n-1} + g\sum_{n=0}^{\infty} b_n t^n = 0$$

The standard trick at this point is to rename the index of summation in each sum so that the powers of t are all the same; this yields

$$\sum_{n=0}^{\infty}\{vb_{n+1}n(n+1) - \frac{1}{2}ab_n n(n-1) + 2vb_{n+1}(n+1) - 2ab_n n + gb_n\}\, t^n = 0.$$

Now it is possible to solve for b_{n+1} since the uniqueness of the power-series expansion implies that $\sum\{---\}\, t^n = 0 \Rightarrow \{---\} = 0$. Thus:

$$vb_{n+1}\{n(n+1) + 2(n+1)\} + b_n[g - 2an - \frac{1}{2}an(n-1)] = 0.$$

Solving for b_{n+1} gives

$$b_{n+1} = b_n \left[\frac{\frac{a}{2} n (n + 3) - g}{v (n + 1) (n + 2)} \right], \quad n = 0,1,2,3,... \tag{5-2}$$

Suppose the polynomial, whose coefficients are specified by (5-2), terminates. Then it will turn out that these are precisely the solutions that return to the pulley. Strictly speaking, these solutions are not periodic, since they complete only one cycle. But it is convenient to say one "period" is the time it takes the bob to return to the pulley. Termination of the series implies

$$b_{n+1} = 0 \quad \Rightarrow \quad \frac{a}{2} N (N + 3) = g.$$

Plugging in the Atwood acceleration $\frac{\mu - 1}{\mu + 1} g$, and expressing μ in terms of N gives

$$\mu = \frac{(N + 1)(N + 2)}{N^2 + 3N - 2}, \quad N = 1,2,3,... \tag{5-3}$$

the so-called <u>magic mass formula</u>. Notice that as N approaches ∞, the magic mass formula goes to one. The masses are called 'magic' because they return to the pulley. The magic mass formula gives an infinite spectrum of rational mass ratios that always return to the center.

The first few magic masses are listed in the table below.

<u>Magic masses</u>

N	μ	Trajectory Name
1	3	Teardrop
2	3/2	Comma
3	5/4	Figure-8
4	15/13	
5	21/19	
6	14/13	
7	18/17	

The trajectories resulting from the first twelve magic masses are shown on the next page. The numerical solutions for the mass ratios are indicated at the bottom of each frame. The swinging bob is fired out at $\theta_0 = 0.5$ radians from the top of each frame. The specific initial conditions are:

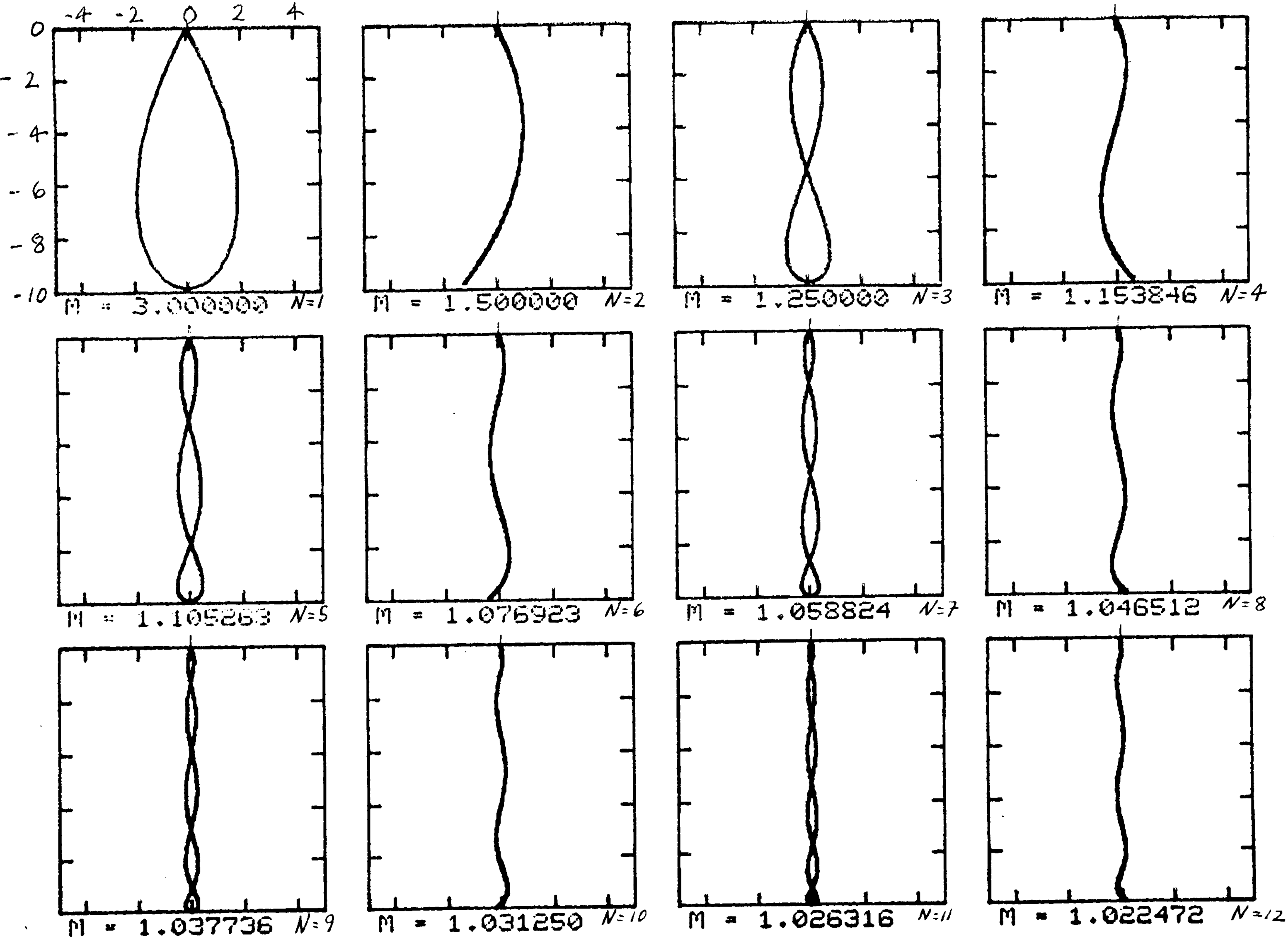
$\mu = M$

$r = 0.01 \qquad \theta_0 = 0.5$ radians

$\dot{r}_0 = v = 10 \quad \dot{\theta}_0 = 0$

$g = 10$

$t: 0 \text{ to } \frac{2v}{a}$

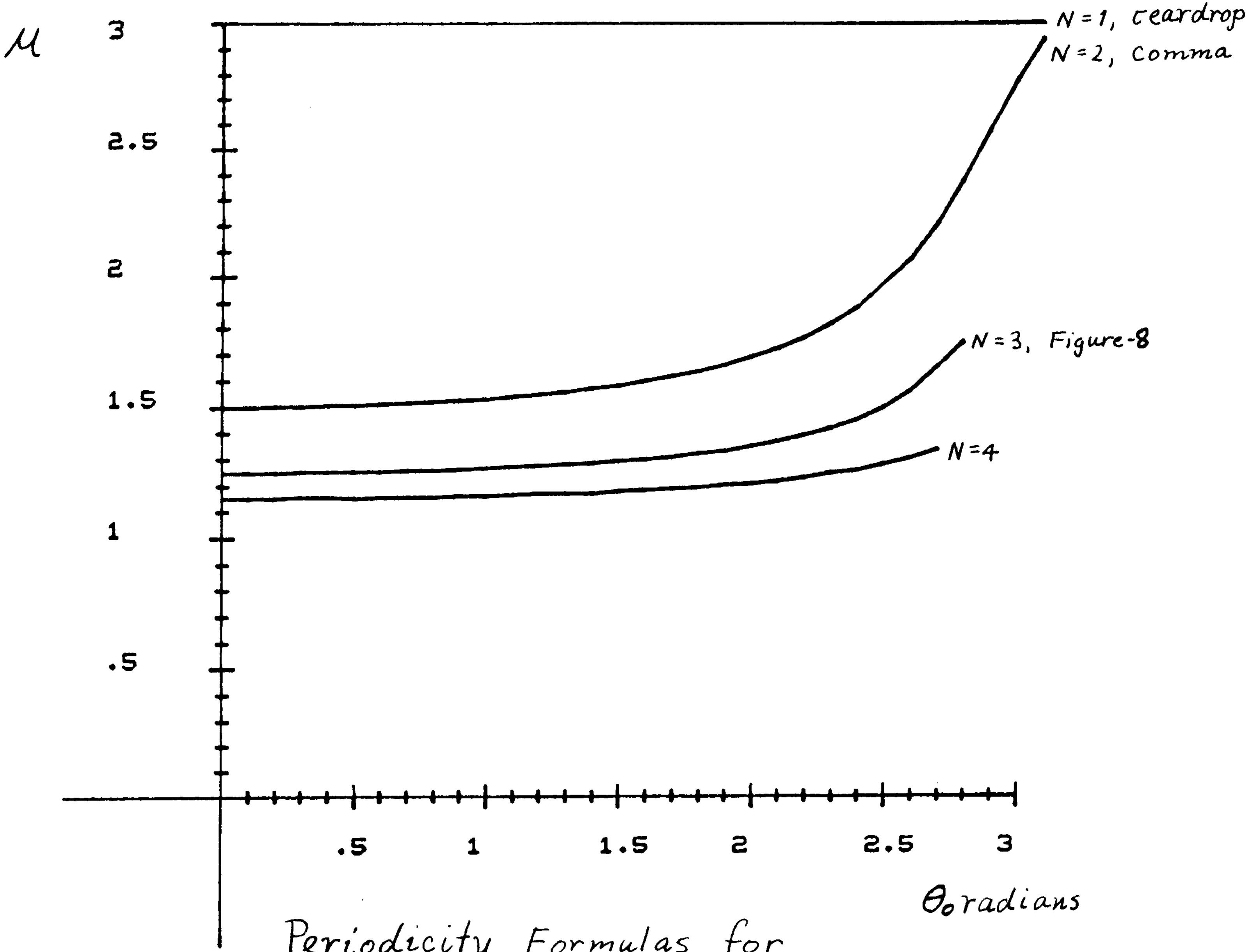

-4 -2 0 2 4
0
-2
-4
-6
-8
-10
M = 3.000000 N=1
M = 1.500000 N=2
M = 1.250000 N=3
M = 1.153846 N=4
M = 1.105263 N=5
M = 1.076923 N=6
M = 1.058824 N=7
M = 1.046512 N=8
M = 1.037736 N=9
M = 1.031250 N=10
M = 1.026316 N=11
M = 1.022472 N=12

The pictures show the trajectory of the bob as it goes away from, and then winds back to the origin. The pattern is similar to that seen in section I.B, Chapter III. As μ approaches one, the solutions are alternatively symmetric and asymmetric about the vertical axis. Like the Smile, the Teardrop crosses the vertical ($\theta = 0$) axis once in one period; the Comma crosses twice, the Figure-8 three times, and so on. This is the same pattern exhibited by the periodic solutions that began from rest in Chapter III.

Of course, the magic mass formula is a first-order result; it is only valid for small angles. There must be a periodicity formula valid for any angle for each type of trajectory that returns to the pulley. These periodicity formulas consist of the particular magic mass plus an additional mass term which is a function of θ_0. Periodicity formulas constructed by numerical methods are given in the next table. The periodicity formula for Teardrops is striking; evidently some masses are more magic than others!

Atwood Periodicity Conditions

Angle	Teardrop	Comma	Figure-8	N = 4
0.1	3.0	1.500	1.250	1.154
0.2	3.0	1.501	1.251	1.154
0.3	3.0	1.502	1.252	1.155
0.4	3.0	1.505	1.254	1.155
0.5	3.0	1.507	1.255	1.152
0.6	3.0	1.511	1.258	1.157
0.7	3.0	1.515	1.260	1.157
0.8	3.0	1.520	1.263	1.158
0.9	3.0	1.525	1.266	1.161
1.0	3.0	1.532	1.270	1.163
1.1	3.0	1.539	1.275	1.166
1.2	3.0	1.548	1.280	1.168
1.3	3.0	1.558	1.286	1.171
1.4	3.0	1.570	1.293	1.171
1.5	3.0	1.583	1.301	1.179
1.6	3.0	1.598	1.310	1.182
1.7	3.0	1.616	1.321	1.188
1.8	3.0	1.637	1.334	1.194
1.9	3.0	1.661	1.349	1.201
2.0	3.0	1.690	1.367	1.209
2.1	3.0	1.725	1.390	1.218
2.2	3.0	1.767	1.418	1.229
2.3	3.0	1.819	1.453	1.250
2.4	3.0	1.883	1.500	1.259
2.6	3.0	2.071	1.564	1.279
2.7	3.0	2.208	1.656	1.305
2.8	3.0	2.381	1.749	1.338
2.9	3.0	2.582		
3.0	3.0	2.780		
3.1	3.0	2.944		



Periodicity Formulas for Almost Atwood's Solutions

II. Teardrops: $\mu = 3$

Solutions that are outwardbound scale in v, the initial radial velocity; i.e., the shape of the trajectory is the same no matter how quickly the bob is fired out. Therefore, Teardrops have the amazing property that if $\mu = 3$, then for any starting angle and velocity, the swinging mass always follows a symmetrical "teardrop trajectory" and returns to the pulley. Furthermore, the numerical studies indicate that the period for any Teardrop is the same as that of an Atwood's machine,

$$T = \frac{2v}{a} = \frac{2v}{g}\frac{\mu + 1}{\mu - 1} = \frac{4v}{g}. \qquad (5\text{-}4)$$

It is easy to obtain a first-order approximation to the Teardrop trajectory by applying the initial conditions to the terminating polynomial given in the power-series solution. The first-order Teardrop can then be used to obtain the second-order correction. A ladder of approximate solutions, which converge to the true solution, is built up in this way. The actual computation is similar to the method used in obtaining the Smile solutions.

The calculations are very involved. In this section an outline of the computation is given along with the conclusion.

The zeroth-order Teardrop is simply an Atwood's machine.

Zeroth-order Teardrop:

$$\theta = 0;$$

$$r = vt - \frac{1}{2}\frac{\mu - 1}{\mu + 1} g t^2$$

To first-order in θ, the radial equation is unchanged; however, the angular equation is determined by applying the initial conditions to the power-series whose coefficients are given by equation (5-2).

<u>First-order Teardrop</u>: $\mu = 3$

$$\theta_1 = \theta_0 (1 - \frac{gt}{2v});$$

$$r_1 = vt (1 - \frac{gt}{4v}).$$

The first-order solution is now used to find a third-order solution. This is done in two steps:

(1) Plug the first-order Teardrop into the radial equation, and expand to second-order in θ_0. Do not assume $\mu = 3$; rather, set

$$\mu = 3 + \beta \theta_0^2.$$

Obtain a solution of the form

$$r_2(t) = r_1(t) + \theta_0^2(??).$$

Demand $r(0) = 0$; $\dot{r}(0) = v$.

(2) Put $r_2(t)$ into the angular equation, and expand to third-order in θ_0. Obtain solution in the form

$$\theta_3 = \theta_1 + \theta_0^3 (A + Bt + Ct^2 + Dt^3).$$

Demand $\theta(0) = \theta_0$, so $A = 0$, and hence

$$\theta_3 = \theta_1 + \theta_0^3 t (B + Ct + Dt^2).$$

Find A, B, C, D, and β.

<u>Conclusion</u>: Put this value of β into the expression for r_2. You now have $r(t)$, $\theta(t)$, and μ for a teardrop, correct to third-order in θ_0.

Anyone daring enough to carry out this calculation will discover that there exists a dimensionless set of variables that is easy to work with. These variables are defined in the *third*-order Teardrop solution.

Third-order Teardrop: $\mu = 3$

Let $z = \frac{gt}{4v}$ and

$w = z(1 - z)$,

$w' = 1 - 2z$

then the third-order Teardrop is

$$r(t) = \frac{4v^2}{g} w [1 - \frac{\theta_0^2}{4} w],$$
$$\theta(t) = \theta_0 w' [1 + \frac{\theta_0^2}{3} w].$$

The third-order Teardrop is next used to calculate a fifth-order Teardrop. Needless to say, the computation is quite long. The solution, though, is relatively simple and there is still no correction to μ in this order.

Fifth-order Teardrop: $\mu = 3$

$$z = \frac{gt}{4v} \quad (5-5)$$

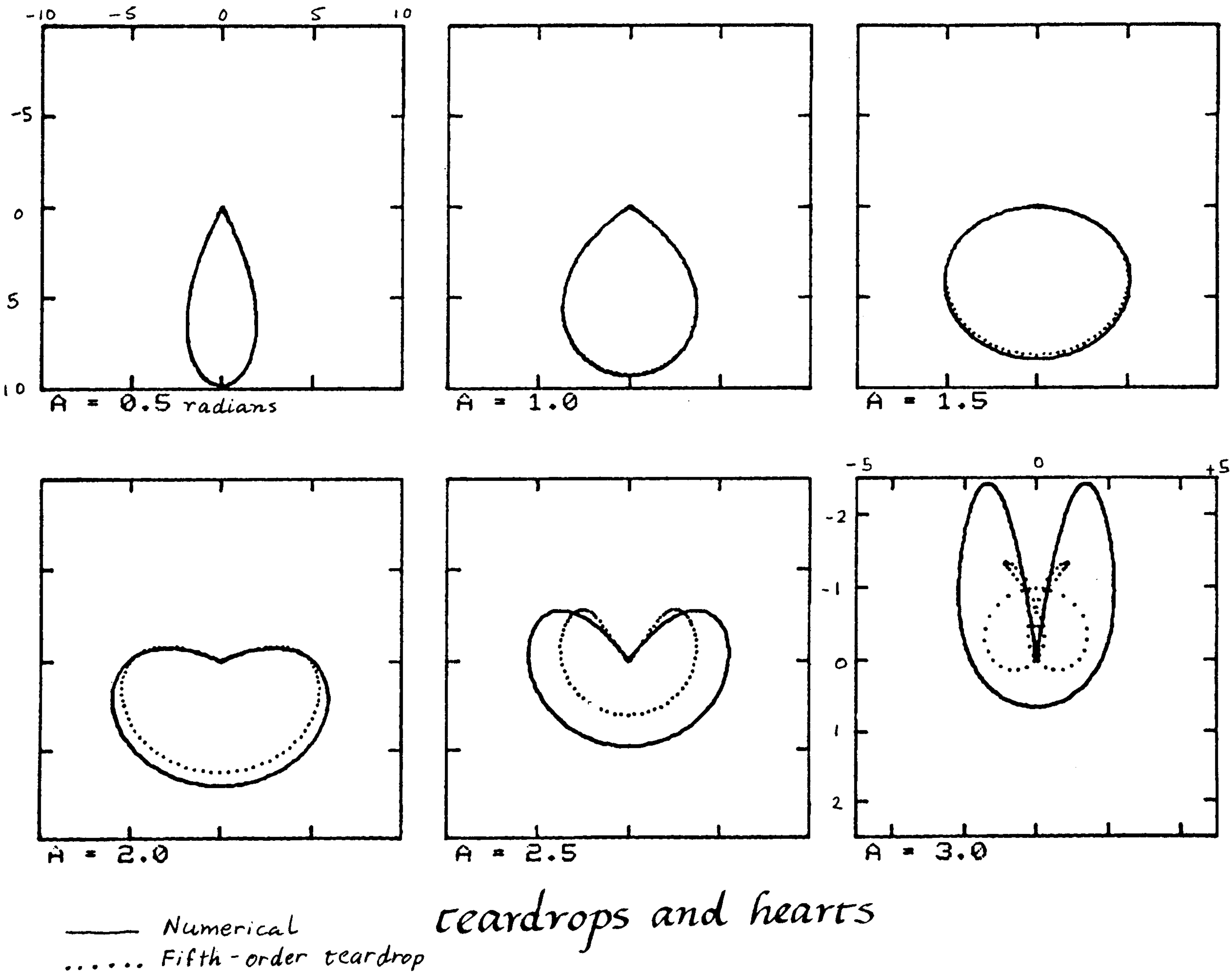
$$w = z(1 - z) \quad (5-6)$$

$$w' = 1 - 2z \quad (5-7)$$

$$r(t) = \frac{4v^2}{g} w [1 - \frac{1}{4}\theta_0^2 w - \frac{1}{8}\theta_0^4 w (w - \frac{1}{6})] \quad (5-8)$$

$$\theta(t) = \theta_0 w' [1 + \frac{1}{3}\theta_0^2 w + \frac{1}{5}\theta_0^4 w (w - \frac{1}{12})]. \quad (5-9)$$

The next page shows a comparison between the numerical solutions and the fifth-order Teardrops. The initial starting angle is indicated at the bottom of each frame in radians. As you can see, for large angles a Teardrop becomes a Heart. The agreement is quite good even for large angles.



teardrops and hearts

III. Commas: $\mu = \frac{3}{2}$

The periodicity formulas constructed for the first four Atwood's type solutions are almost constant until $\theta_0 = 1.5$ radians. This suggests that the first correction term to any of the magic masses is larger than θ_0^2. During the investigation of this conjecture, the Comma trajectories was approximated by the method outlined in the previous section. It is found that even for Commas, there is no correction of μ of order θ_0^2. The third-order Comma solution is

Third-order Comma: $\mu = \frac{3}{2}$

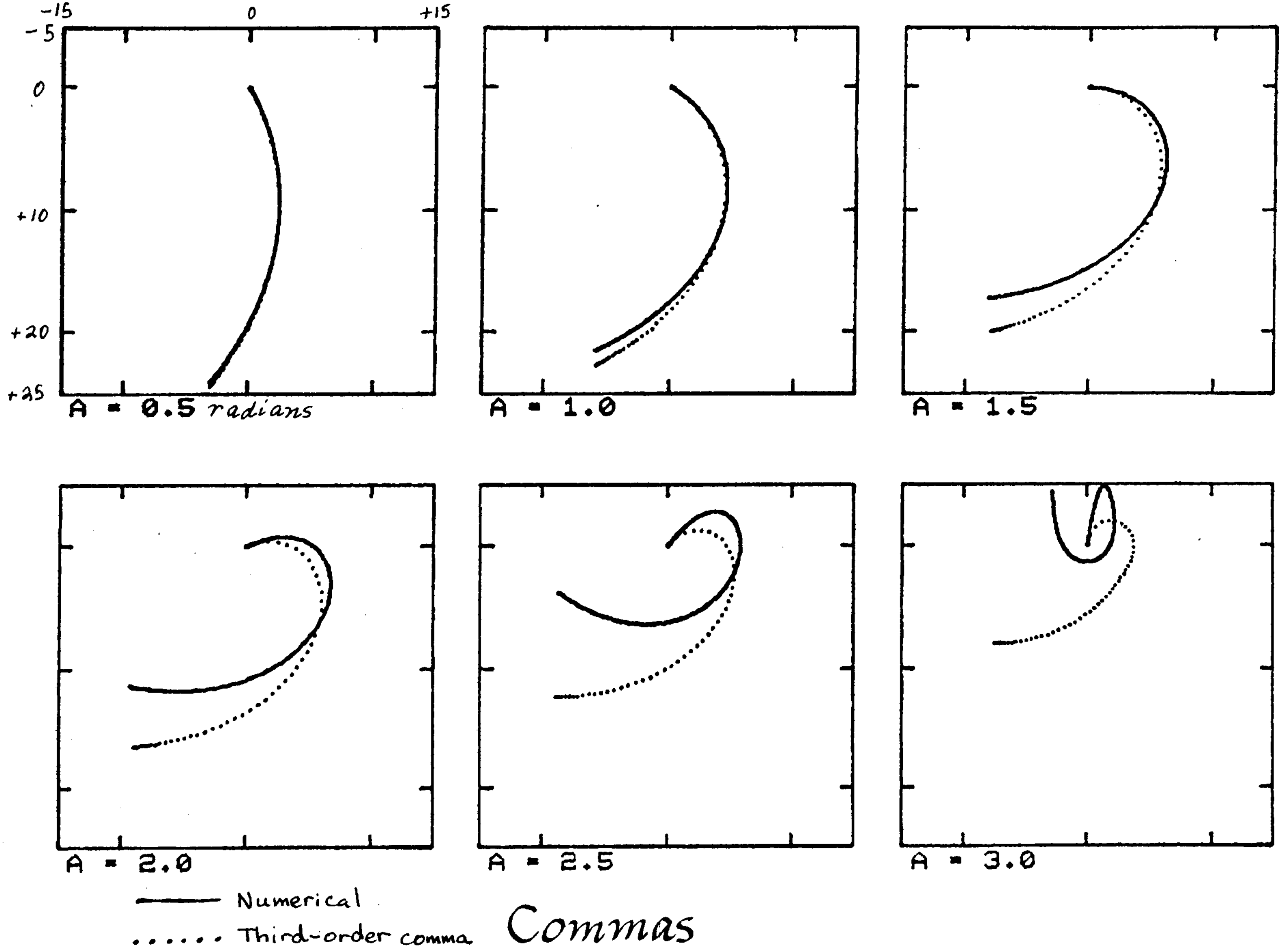
$$\theta(t) = \theta_0 (1 - 5w) \tag{5-10}$$

$$r(t) = \frac{10v^2}{g} w[1 - \theta_0^2 w(1 - 3w)] \tag{5-11}$$

$$w = z(1 - z) \tag{5-12}$$

$$z = \frac{gt}{10v}. \tag{5-13}$$

A comparison between the perturbative and numerical solutions is given on the next page.



-15
0
+15
-5
0
+10
+20
+25
A = 0.5 radians
A = 1.0
A = 1.5
A = 2.0
A = 2.5
A = 3.0
Numerical
Third-order comma

Commas

CHAPTER VI. Smiles Exist

Q: How many mathematicians does it take to screw in a light bulb ?

A: $n \in Z^+$

"There is always an easy solution to every human problem — neat, plausible and wrong."

H.L. Mencken

I. Motivation

The argument presented here, to show the existence of periodic motions, is based on the following simple observation.

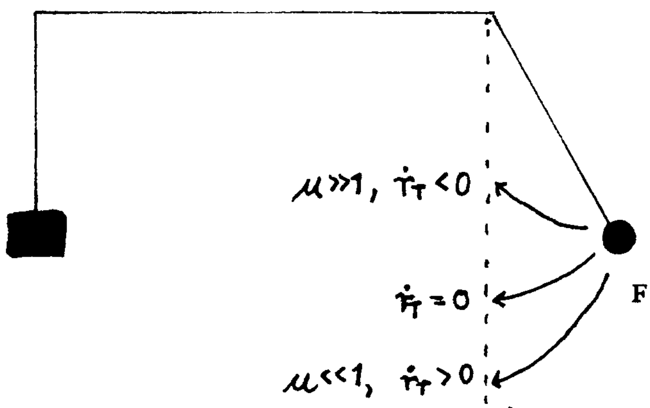


Fig. 6-1

Trajectories for large and small μ; $\dot{r}_T$ denotes the radial velocity at $\theta(T) = 0$.

For motion that begins from rest, the radial velocity when the bob crosses the vertical axis, is positive when μ is very small. On the other hand, if μ is very large then the radial velocity is negative as the bob crosses the vertical axis (cf. Fig. 6-1). As μ varies continuously between the two extremes, there must occur an intermediate μ such that the resulting trajectory crosses the vertical axis with zero radial velocity.

But if the bob begins on the other side of the vertical axis, with

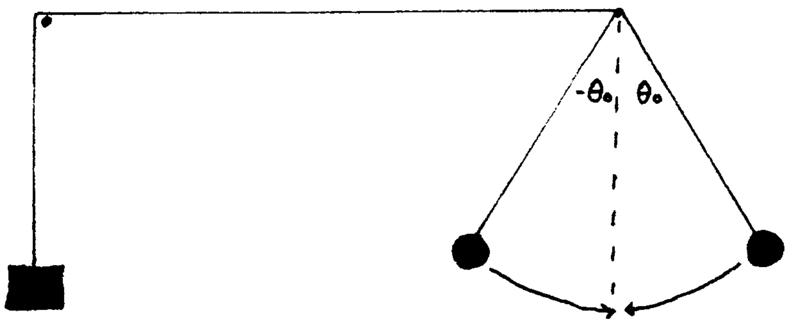


Fig. 6-2

the same initial conditions, then a similar trajectory will result. Gluing these two solutions together gives a periodic trajectory. Hence, any motion that starts from rest and crosses the vertical axis with zero radial velocity is periodic. QED

In the qualitative argument just presented, four plausible statements are made which will be proved more rigorously (i.e. directly from the equations of motion) in the next section: to wit,

(a) The trajectory does cross $\theta = 0$.
(b) For large enough μ it crosses with $\dot{r} < 0$.
(c) For small enough μ it crosses with $\dot{r} > 0$.
(d) If it crosses with $\dot{r} = 0$, the trajectory is periodic.

II. Proof

Throughout this proof, set $g = 1$ and $r_0 = 1$. This involves no loss of generality as mentioned in Chapter II. Further, consider only motions that start from rest with $0 \leq \theta_0 < \frac{\pi}{2}$.

Define T as the time required for the first zero crossing; i.e., the first time $\theta(t) = 0$. In general, T is a function of θ_0 and μ. For Smiles, T is one quarter of the period.

$T(\theta_0, \mu)$ is assumed to be continuous for the motions considered. Given that $T(\theta_0, \mu)$ is continuous, it will now be argued that there exist periodic motions that are symmetric about the vertical axis.

LEMMA 6.1

$\theta(t)$ is monotonically decreasing, for $0 < t < T$.

Proof:

$$\frac{d}{dt}(r^2\dot{\theta}) = -r\sin\theta \Rightarrow \dot{\theta} = \frac{-1}{r^2}\int_0^{\tau} r\sin\theta\, dt.$$

Now, $r\sin\theta > 0$ for $0 < \theta < \pi$ so

$$\dot{\theta}(t) \leq 0$$

until $\theta(t) = 0$, since $\dot{\theta}(0) = 0$. QED

A useful equality is given by the next Lemma.

LEMMA 6.2

$$\frac{1 - \ddot{y}}{y} = \frac{\mu}{\mu + 1}\frac{1}{r}\,[1 + \cos\theta + r\dot{\theta}^2]$$

Proof: Because $y = r\cos\theta$, it follows that

$$\ddot{y} = \ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta.$$

Plugging in the radial and angular equations, it is possible (see Lemma 3.2) to rewrite $\ddot{y}$ as:

$$\ddot{y} = 1 - \frac{\mu}{\mu + 1}\cos\theta\,[1 + \cos\theta + r\dot{\theta}^2].$$

Therefore,

$$\frac{1 - \ddot{y}}{y} = \frac{\frac{\mu}{\mu + 1}\cos\theta\,[1 + \cos\theta + r\dot{\theta}^2]}{r\cos\theta}.$$

$$= \frac{\mu}{\mu + 1}\frac{1}{r}\,[1 + \cos\theta + r\dot{\theta}^2].\ \text{QED}$$

THEOREM 6.1

There exists a T such that $\theta(T) = 0$.

Proof: Equation (2-12) — the angular equation in Cartesian coordinates — reads:

$$\ddot{x} = \frac{-(1 - \ddot{y})}{y} x = \frac{-\mu}{\mu + 1} \frac{1}{r} [1 + \cos\theta + r\dot{\theta}^2] < 0$$

until $\theta(t) = 0$. Since $x_0 > 0$ and $\dot{x}_0 = 0$, there exists a T such that $x(T) = 0$, and hence $\theta(T) = 0$.

THEOREM 6.2

Let $\ddot{x} = -\omega^2(t)\,x$ with $x(0) = x_0$; $\dot{x}(0) = 0$. Assume $\omega^2(t) \geq \omega_0^2 \geq 0$; ω_0 is a constant. Then the time T required to reach $x(T) = 0$ satisfies:

$$T \leq \frac{\pi}{2\omega_0}$$

Proof:* Until $x = 0$, examine

$$\frac{d}{dt}\left(\frac{\dot{x}}{x}\right) = \frac{x\ddot{x} - \dot{x}^2}{x^2} \leq -(\omega_0^2 + \frac{\dot{x}^2}{x^2}), \text{ then}$$

$$\frac{\frac{d}{dt}\left(\frac{\dot{x}}{x}\right)}{\omega_0^2 + \frac{\dot{x}^2}{x^2}} \leq -1,$$

$$\frac{d}{dt}\left[\arctan\left(\frac{\dot{x}}{\omega_0 x}\right)\right] \leq -\omega_0.$$

Integration yields

$$\frac{\dot{x}}{\omega_0 x} \leq -\tan(\omega_0 t),$$

$$\frac{d}{dt}[\log(x)] \leq \frac{d}{dt}[\log(\cos(\omega_0 t)].$$

* Proof by Ray Mayer.

Integration again results in

$$\log \frac{x(t)}{x_0} \leq \log[\cos(\omega_0 t)], \text{ or}$$

$$x(t) \leq x_0 \cos(t\omega_0). \text{ QED}$$

What this says is that an oscillator whose "frequency" $\omega(t)$ is a function of time, but always <u>less</u> than some constant ω_0, will stay <u>under</u> the curve for the SHO with frequency ω_0:

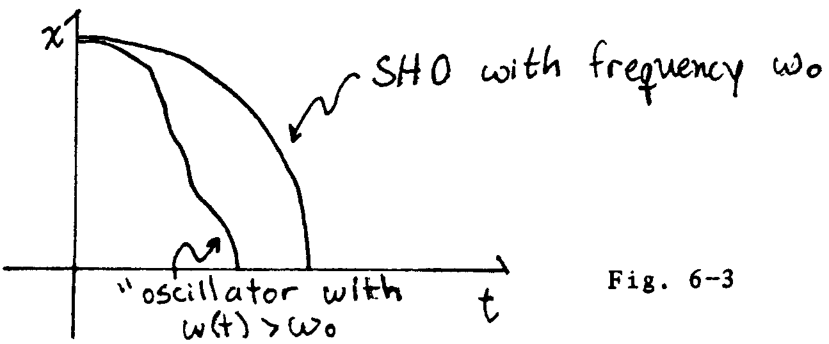


Fig. 6-3

Consequently, it must hit the t axis earlier than the SHO, which crosses at $T = \frac{1}{4}\frac{2\pi}{\omega_0}$. QED

REMARK 6.1

In this particular case, a lower bound for the "frequency", ω, can be obtained from equation (2-12),

$$\ddot{x} = \frac{-(1 - \ddot{y})}{y} x = -\omega^2(t)\, x \quad \text{where } \omega^2(t) = \frac{1 - \ddot{y}}{y}.$$

So a lower bound, ω_0^2, is

$$\omega_0^2 = \frac{\mu}{\mu + 1} \frac{1}{r_{max}} [1 + \cos\theta_0] \leq \omega^2(t).$$

In theorem 3.1, it is shown that r_{max} is bounded by

$$r_{max} \leq \frac{\mu - \cos\theta_0}{\mu - 1}$$

when $\mu > 1$ and the motion starts from rest. Combining this result with theorem 6.2 results in an upper bound on the time, T_{max}, it takes for the first zero crossing:

$$T \leq \frac{\pi}{2}\frac{1}{\omega_0} = \frac{\pi}{2}\sqrt{\frac{\mu + 1}{\mu(\mu - 1)}\frac{\mu - \cos\theta_0}{1 + \cos\theta_0}},$$

for $\mu > 1$. This bound is essential for the next theorem.

THEOREM 6.3

For some $\mu > 1$, there is a trajectory such that $\dot{r}(T) \leq 0$.

Proof: If $\ddot{r} \leq 0$ on $0 \leq t \leq T$ then $\dot{r}(T) \leq 0$. To show $\ddot{r}$ is always negative on this interval, a lower and upper bound is needed for r. A lower bound is obtained from considering the minimal radial acceleration:

$$\ddot{r} = \frac{1}{1 + \mu}(r\dot{\theta}^2 + \cos\theta - \mu) \Rightarrow r \geq 1 - \frac{1}{2}\frac{\mu - \cos\theta_0}{\mu + 1}t^2$$

Again, the upper bound for r given by Theorem 3.1 for $\mu > 1$ will be sufficient.

The condition $\ddot{r} \leq 0$ is equivalent to:

$$r\dot{\theta}^2 + \cos\theta_0 \leq \mu.$$

An upper bound for $r\dot{\theta}^2$ can be obtained from the angular equation,

$$r\dot{\theta}^2 = \frac{1}{r^3}\left[\int_0^T r\sin\theta\, d\tau\right]^2 \leq \frac{1}{r_{min}^3}[r_{max}^2 \sin^2\theta_0\, T^2]$$

where

$$r_{max} = \frac{\mu - \cos\theta_0}{\mu - 1}$$

$$T_{max}^2 = \frac{\pi^2}{4}\frac{\mu + 1}{\mu}\frac{r_{max}}{1 + \cos\theta_0}$$

$$r_{min} = 1 - \frac{1}{2}\,\frac{\mu - \cos\theta_0}{\mu + 1}\,T_{max}^2$$

For the case $\mu = 2$, and $\theta_0 = 0.1$ radians, the left hand side of the inequality is about 1.6, which is less than 2. QED

THEOREM 6.4

For some $\mu < 1$, there is a trajectory such that $\dot{r}(T) \geq 0$.

Proof: Let $\theta_0 \leq \frac{\pi}{4}$ and $\mu = \frac{\sqrt{2}}{2}$, then

$$(1 + \mu)\ddot{r} = r\dot{\theta}^2 + \cos\theta - \mu \geq 0$$

until $\theta = 0$. Therefore, $\ddot{r} \geq 0$ on the entire interval so $\dot{r}(t) > 0$ at $\theta(T) = 0$. QED

The hypothesis of theorem 6.5 follows from theorems 6.3 and 6.4, and the assumption of continuity. It shows how the solutions are to be 'glued' together.

THEOREM 6.5

If $\dot{r}(0) = \dot{\theta}(0) = 0$, and for some $T > 0$, $\theta(T) = \dot{r}(T) = 0$, then the motion $\{r(t), \theta(t)\}$ is periodic.

Proof: Let $n = [t/T]$, greatest integer in t/T. Let $\tau = t - nT$. By hypothesis, r and θ are defined on $(0,T)$. Define:

$$\bar{r}(t) = \begin{cases} r(\tau) & ; n = 0 \pmod 4 \\ r(T - \tau) & ; n = 1 \pmod 4 \\ r(\tau) & ; n = 2 \pmod 4 \\ r(T - \tau) & ; n = 3 \pmod 4 \end{cases}$$

$$\bar{\theta}(t) = \begin{cases} \theta(\tau) & ; n = 0 \pmod 4 \\ -\theta(T - \tau) & ; n = 1 \pmod 4 \\ -\theta(\tau) & ; n = 2 \pmod 4 \\ \theta(T - \tau) & ; n = 3 \pmod 4 \end{cases}$$

Note that $\bar{r}$ and $\bar{\theta}$ are now defined for all t. Furthermore, by invariance of motion under,

$$\theta \longrightarrow \pm\theta(\pm t)$$

$$r \longrightarrow r(\pm t)$$

$\bar{r}$, $\bar{\theta}$ and $\dot{\bar{r}}$, $\dot{\bar{\theta}}$ are continuous. $\bar{r}$, $\bar{\theta}$ are solutions for all $t > 0$ and are clearly periodic. QED

III. Poincaré's Geometric Theorem

The argument just presented gives one little insight into the structure of the periodic motions. For instance, it fails to make use of the observation that motion starting from rest is periodic if and only if there exist two rest points (at some $t > 0$, $\dot{r} = \dot{\theta} = 0$).

It is possible that an alternative proof is furnished by "Poincaré's last geometric theorem", which applies specifically to problems with two degrees of freedom. This theorem reduces the existence of periodic motions to a study of the fixed points of a "surface of section", in particular the fixed points of the mapping of the annulus (a typical surface of section) to itself. A modern exposition of this theorem is found in Appendix 9 of V.I. Arnold's Mathematical Methods of Classical Mechanics. The proof depends upon the construction of a two-dimensional "surface of section" which is described in Appendix 7.*

This theorem has only been applied to a few concrete dynamical systems (e.g. the restricted three body problem). So a proof along these lines would be of substantial mathematical and physical interest.

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* V.I. Arnold, Mathematical Methods of Classical Mechanics (Springer-Verlag, New York 1978).

CHAPTER VII. Future Research

"there is nothing more delightful then discovering truths for one's self".

H.A. Roland

"a time to weep and a time to laugh".

Ecclesiastes, c. 200 B.C.

I. Dynamical Systems Theory

The original aim of this study was to apply the methods and theorems of dynamical systems theory to a simple physical problem. Dynamical systems theory had its origins in the problems of celestial mechanics, but in modern times it has blossomed into an independent mathematical discipline.

I was encouraged to undertake such a program by the early success of G. D. Birkhoff in analyzing a general potential system with two degrees of freedom. I was particularly struck by the similarity between the diversity of motions indicated by the numerical studies of SAM and the conclusion reached by G. D. Birkhoff in his paper "Surface transformations and their dynamical applications":*

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* George David Birkhoff, Collected Mathematical Papers Vol. II (New York: Dover, 1950) p. 119.

"The varying degree of definiteness of the results above obtained for dynamical systems is striking. The catalogue of types of motion according to their degree of simplicity appears to run as follows: ordinary periodic motions, biperiodic motions representable analytically by convergent trigonometric series in two arguments, triperiodic motions representable by three arguments; motions asymptotic to periodic motions of the hyperbolic type, motions asymptotic to periodic motions of elliptic type and of the other types just referred to; recurrent motions of the biperiodic or triperiodic type (not representable by convergent trigonometric series); recurrent motions of discontinuous type; motions asymptotic to recurrent motions of these new types (or to sets of isomorphic recurrent motions); <u>special</u> motions (i.e. not passing near all phases for both lim $t = +\infty$ and lim $t = -\infty$) not of above types; <u>general</u> motions.

The degree of definiteness attained has varied with the analytic instruments at hand, and will probably be found to correspond to the nature of the case, at least unless entirely new analytic instruments are discovered.

The remarkable diversity and complexity of structure possible in dynamical systems with two degrees of freedom is likely to stand permanently in the way of approach to any definitive form for the theory of such systems. As has appeared above, many of the most vital questions are still without an answer. Progress with these questions and progress with the theory of the conservative transformations T which we have studied will go hand in hand."

Of course, many advances have been made since Birkhoff's time, in particular, "the stability of positions of equilibrium and periodic solutions of conservative systems with two degrees of freedom has been proved in the so-called elliptic case."† No doubt, the applications of such results would provide insight into both SAM, and general problems arising in the theory of dynamical systems with two degrees of freedom.

<u>II</u>. <u>Orbit</u> <u>Equation</u>

A differential orbit equation is obtained by eliminating time from the equations of motion and expressing r in terms of θ. It is listed here as a reference for future research. If

$$r' = \frac{dr}{d\theta} = \frac{\dot{r}}{\dot{\theta}}, \quad (\dot{\theta} \neq 0)$$

† V.I. Arnold, Russ. Math. Surv. 18, 85 (1963).

then the elimination of $\dot{\theta}^2$ by means of the energy equation leads to the orbit equation

$$r'' = \frac{r}{1 + \mu} + \frac{2r'^2}{r} + \left[\frac{\cos\theta - \mu}{1 + \mu} + \frac{r\sin\theta}{r} r'\right]\left[\frac{(1 + \mu)r'^2 + r^2}{2}\right]\left[\frac{1}{E + r(\cos\theta - \mu)}\right]$$

III. Exercise

A lovely solution is obtained by setting $g = 0$ in the equations of motion.

Problem 7.1

When the swinging mass 'whips' around the pulley, g can be ignored in the equations of motion so set $g = 0$ in the angular and radial equations. Develop an differential orbit equation by eliminating t, i.e. express r in terms of θ. Solve this second order ODE. What is special about $\mu = 3$ in the solution? Assuming that the periodicity formula is monotonically increasing, prove for Teardrops that $f(\theta_0) = 3$ for any θ_0. What is the relation between Teardrops and Smiles, based on this solution. Is there a continuous transformation from the Smiles to the Teardrops?

APPENDIX 1: Numerical Methods

All numerical intergations are accomplished with Ode,* a research tool of remarkable utility. A typical Ode program looks like:

```
# SAM: simulation of a Swinging Atwood's Machine

# G is the gravitational constant
G = 10
# m is the swinging mass, M is the nonswinging mass
m = 1
M = 3

# initial conditions:
# a is starting angle in radians
# adot is initial angular velocity
# r is the starting length of the string
# rdot is the initial radial velocity
a    = 2*PI
adot = 0
r    = 0.01
rdot = 10

# the angular equation
a'    = adot
adot' = -G/r * (2*rdot + sin(a))
# the radial equation
r'    = rdot
rdot' = 1/(m+M) * (m*r*adot 2 + m*G*cos(a) - M*G)

# the print statement
print a, r
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A variable step size Runge-Kutta-Fehlberg (RKF) algorithm of order five is employed in all solutions. The maximal relative single step error never exceeded 1e-8 in any of the data listed; the relative single step error is held below 1e-14 during the construction of some of the

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For a complete discussion of the numerical and linguistic aspects of Ode see: Nick Tufillaro and Graham A. Ross, Ode User's Manual, Reed College Academic Computer Center, 1981. An on-line version of the manual is available in /usr/doc. To get a copy say 'run ode'.

periodicity formulas.

The numerical construction of the periodicity formulas is essentially a boundary value problem. They are constructed with the 'shooting method'; a simple bisection algorithm is used to home in on the desired mass ratio. The current version of Ode is not designed to handle a boundary value problem since it lacks decision statements. In order to overcome this short fall, Ode is embedded within the C programming language so that, as far as the parent program is concerned, Ode simply looks like a subroutine which generates the trajectories. This is accomplished by opening pipes to and from Ode, so that the input and output to Ode are controlled by a parent C program. Quite novel programs were also developed which could automatically seek out and home in on periodic solutions in any specified mass range, but because of floating point hardware problems these investigations were never completed.

The total energy is monitored during the entire numerical solution as a check on the numerical method. The energy is found to vary by less than one percent even over hundreds of oscillations.

At last report, these programs are located somewhere near /u/s/tufil/the/prg.