Definitions of Chaos and Measuring its Characteristics

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Abstract

The topic of this symposium is Optical Chaos. It is essential for those working in this field to know how to recognize, identify and characterize chaotic behavior and how to distinguish it from noise. General principles, definitions of terminology, and some brief review and commentary on the current status of techniques for measuring chaos are presented.

Introduction

Deterministic systems are those for which there are definite prescriptions for all their future behavior. The future is completely determined by the present state of the system. Contrary to common teaching, it is possible for deterministic systems to evolve over long times in an irregular fashion, the details of which are such that the specific future is unpredictable in practice because of uncertainties in knowing the initial conditions. It is to this type of deterministic behavior that the term chaos is applied. Remarkably, not only do these deterministic irregular solutions exist, but often they can be distinguished from stochastically (thermodynamically) irregular behavior resulting from the combination of an extremely large number of independent random variable. Of course, most deterministic physical systems are not completely isolated from random perturbations (in optics we must often deal with inescapable random spontaneous emission), so in order for deterministic irregularity (chaos) to be a useful concept, we must be able to show that there are some features of at least some chaotic systems that are robust enough to persist in the presence of weak noise perturbations. It is in this context that we are able now to say (and believe) that chaotic behavior has been observed and that it has been distinguished from random behavior in a number of systems including a variety of optical experiments.

In this overview we will focus on dissipative dynamical systems, as most optical systems are of this type. Chaos is also found in conservative (often called Hamiltonian) systems but its character will not be reviewed here. Several references provide an entry into this field[1-3].

Many other overviews and reviews of chaotic dynamics should be consulted for details which go beyond the discussion presented here[4-10]. For optical systems, discussions of chaos have been provided in the reviews by Milonni, Shih and Ackerhalt[11], and Biswas and Harrison[12] and in compendia of research reports[14-16].

Types of solutions of nonlinear dynamical systems

We can focus our attention by considering first the simplest laser model[17-19], the equations for a single mode field interacting with a collection of homogeneously-broadened, two-level atoms, a set of equations which Haken[20] has shown are isomorphic to the Lorenz equations used in some of the earliest efforts to describe convective turbulence,

$$E = -\kappa E - \kappa AE, \tag{1}$$

$$P = -\gamma_\perp (P\{1 + i\delta_{AC}\} + DE), \tag{2}$$

$$D = -\gamma_\parallel [(D-1) - \tfrac{1}{2}(P^*E + PE^*)], \tag{3}$$

Where E, P, and D are the slowly varying amplitudes of the electric field, atomic polarization, and atomic population inversion, respectively, with respect to the cavity resonance frequency while $\kappa, \gamma_\perp$, and $\gamma_\parallel$ are their respective relaxation rates. The parameter A governs the density of the inverted medium and is normalized so that for $A > 1+\tilde{\Delta}^2$, there is a nonzero constant intensity solution. δ_{AC} is the detuning of the atomic resonance frequency from the cavity resonance frequency, and $\Delta(\Delta=\delta_{AC}/(1+\tilde{\kappa}))$ is the resultant detuning of the frequency of the single-mode steady state solution for laser from the atomic resonance frequency. We do not lose any important generality by considering first order differential equations, because higher order systems can be reduced to first order systems by expanding the number of variables. It is crucial, however, that the system be appropriately coupled and nonlinear. Tildes over variables denote division by $\gamma_\perp$.

In order to better understand the nature of chaotic solutions, we first consider other kinds of solutions. "Steady State Solutions" are those found by setting the time derivatives equal to zero. For steady state solutions every variable takes on a constant, time-independent value. For this problem there are two types of steady state solutions. The trivial solutions of E=P=0; D=1 which exists for all parameter values. For $A > 1 + \tilde{\Delta}^2$, a second solution exists given by $|E|^2 = A - (1+\tilde{\Delta}^2)$; $P = -(1-i\tilde{\Delta})E/A$; $D = (1+\tilde{\Delta}^2)/A$. This second solution is the normal lasing solution of constant output intensity. One way to show how these solutions change as some parameter is varied is to construct what is called a "steady state bifurcation diagram" as in Figure 1 where the value of the intensity I $(=|E|^2)$ of different solutions is plotted versus A.

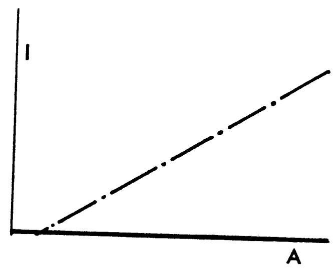

Figure 1: Steady state bifurcation diagram showing I=0 solution and I=A-1 solution for Equations 1-3 as a function of the excitation parameter A.

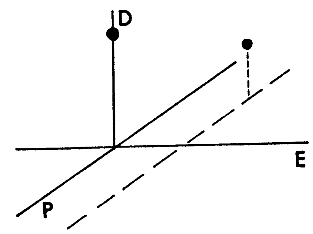

Figure 2: Schematic vectors in the variable space for A=2. The I=0 solution is the trivial point E=P=0, D=1; while the I=A=1 solution is given by E=1, P=-1, D=.5.

Another way to represent the solution for particular parameter values is as a vector in the variable space as shown in Fig 2. Because we are using first order differential equations it is sufficient to represent the system in the "configuration space" (space of all the variables of the system). If we were using equations which were second-order differential equations in time we would have to represent the system in "phase space" (variables and their derivatives).

For certain parameter values, there are periodic solutions. As it is hard to represent solutions in a multivariate space, we often choose the simplification of plotting either one variable in time, a projection of the solution in the full variable space onto a two-dimensional subspace, or labelling the types of solution in some special ways and plotting some particular characteristic (such as the maximum value of the intensity of the periodic solutions) on the bifurcation diagram. The steady state bifurcation diagram becomes a more generalized bifurcation diagram showing all periodic solutions. Some samples of these representations for periodic solutions of the laser equations are shown in Figure 3.

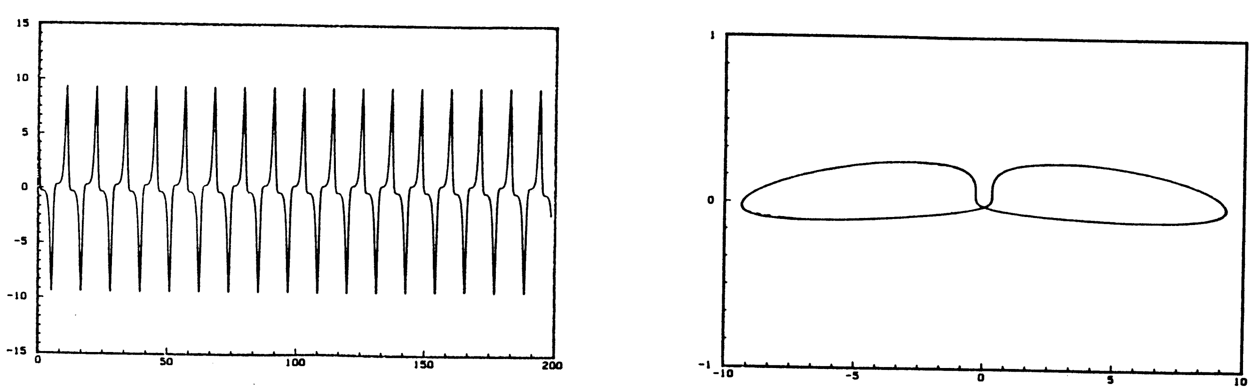

Figure 3: Sample results for solutions of equations 1-3 yielding periodic solutions. a) electric field E versus time; b) D vs. E showing a simple closed loop attractor. Apparent crossing of the trajectory is because this is viewed as projected. Parameter values A=16, γ=.1, κ =3, Δ =0.

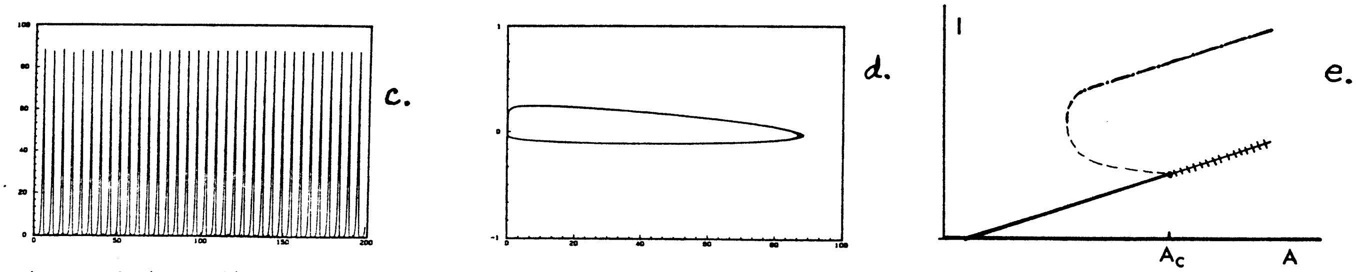

Figure 3 (contd): Characteristics of periodic signal plotted in parts a and b. c) Intensity vs. time, d) D vs. intensity, e) schematic bifurcation diagram showing the steady state solution unstable above A_c and branch of periodic solutions indicated by dash-dot line.

Periodic solutions are not necessarily simple loops unless there are only two variables in the space. When there are three or more variables, periodic solutions may be complicated, multiloop structures which have the particular feature that they close exactly.

Periodic solutions will have distinctive power spectra with a peak at the fundamental oscillation frequency and with smaller peaks at its harmonics if the oscillation is not sinusoidal. If there are multiple loops before the solution repeats, then one can expect a subharmonic given by the fundamental frequency divided by the number of loops. Recalling that a deterministic solution cannot return to a prior value without immediately beginning to repeat, we notice immediately that subharmonics are possible only if there are at least three variables.

"Quasiperiodic solutions" are those which involve two distinct frequencies. If the frequencies f_1 and f_2 are rationally related ($f_1/p = f_2/q$, where p and q are integers) then the solutions may look like a closed spiral. If the frequencies are irrationally related, then the solutions will trace out all of the surface of a torus, obviously indicating that at least three variables are involved. The spectrum will show peaks corresponding to the two frequencies (and their harmonics) and depending on the degree of nonlinearity there will also be peaks at frequencies which are rational linear combinations of the two frequencies.

With these other easily identified types of solutions as background, we show examples of chaotic solutions of the laser equations.

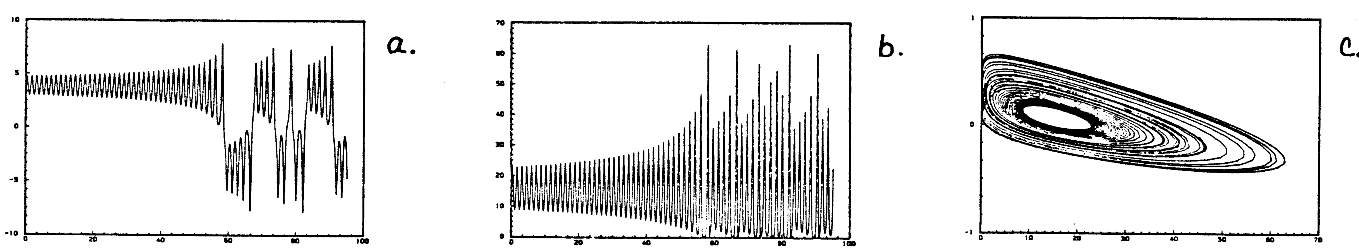

Figure 4: Samples of chaotic solutions of equations 1-3; a) E vs. time, b) intensity vs. time, c) D vs. intensity helping to visualize the strange attractor in projection. Values for A=16, $\gamma = 1.0$, $\Delta = 0$, and $\kappa = 4.0$.

Chaotic solutions have the following features:
1) Irregular time dependence,
2) Broadband power spectra,
3) Exponential divergence of nearby trajectories,
4) Another trajectory arbitrarily close to any part of any trajectory,
5) Occupying a subspace of fractal dimensions.

The chaotic nature of the signal may be the dominant feature of the evolution of the variables, or it may be only a weak perturbation about a steady state or a periodic or quasiperiodic solution. If the peaks of a chaotic signal are added to a bifurcation diagram there is blurring in some regions of the plot, in contrast to the isolated points representing peaks of periodic solutions.

All of these types of solutions are of little importance unless they are stable. Stability refers to the fact that the system returns to the same subset of the variable space for asymptotically long times after a small perturbation. Stable solutions are called attractors. The sensitive dependence of orbits on their initial conditions and the restriction of the solution to a fractal subspace that are characteristic of an attracting yet chaotic solution, cause it to be called a "strange attractor."

It is possible that random noise, suitably filtered for its frequency components can have a spectrum identical to that of a chaotic solution. A power spectrum reveals only the autocorrelation of a time signal from a stationary process which is not specific enough to uniquely determine the signal itself. How then can one tell whether the irregular evolution of a variable has its origin in deterministic processes or stochastic processes? There are both qualitative and quantitative methods.

One qualitative technique is to observe particular sequences in the evolution of the attracting solution as a parameter of the system is changed. For many nonlinear dynamical systems it has been shown that chaos is reached by one of a particular set of sequences of solutions[21]. These universal scenarios include a sequence of period doublings (generating successive subharmonics), a sequence from periodic, to quasiperiodic, to chaotic behavior, or a sequence from periodic, to random interruptions in the periodic behavior which increase in average frequency as the parameter is varied. Unfortunately, many systems reach chaotic behavior by "nongeneric" routes so these scenarios are often not helpful in determining whether the behavior observed is actually chaotic. Furthermore, exact universality of the scenarios is achieved only for particular limits of the parameters which can usually not be reached in real experiments. Often a period doubling sequence is truncated prematurely by noise (or by special dynamical effects) leading to chaos, but it is hard to persuade skeptics that noise can truncate a dynamical sequence and yet not play an equally distracting role in the chaotic evolution that follows. Hence, scenarios can be observed, at best, only approximately and so one must turn to more quantitative ways if one is to have assurance that the broadband spectra arise from chaos. Another method is to plot "return maps" of, for example, each peak versus the previous peak or each period versus the previous period. In this case the idea is to see if there is a particularly limited set of possible pairings of peaks. If the set of points appears to be a fractal set rather than randomly or uniformly distributed this is a strong indicator that the attractor lies on a fractal of low dimensionality.

To quantitatively distinguish noise from chaos, one must measure one or more of the particular properties that are specially properties of chaos. These may include measuring the exponential divergence of trajectories (the Lyapunov exponents), measuring the dimension of a return map or of the attractor itself, or measuring the entropy generated by the solution. Recently other measures have been proposed including a whole spectrum of dimensions and entropies[22,23]. Techniques for each of these measurements have been developed to apply to both numerical solutions and to experimental data.

The measurement of a fractal dimension of an attractor has emerged as the simplest numerical procedure to use to confirm that chaotic behavior underlies a random-looking signal and its associated broadband power spectrum. A widely used technique is that proposed by Grasberger and Procaccia[24]. Examples and discussions of this technique as applied co optical systems can be found elsewhere[25-27]. The measurements are made by collecting a set of values of one variable of the system with successive values equally spaced in time. Due to fortunate results from the topology of nonlinear systems, it can be shown that the topological features of an attractor can be fully reconstructed from the time behavior of this one variable. The reconstruction is achieved by plotting the trajectory in a vector space formed by taking as vectors the sets of sequential values which are delayed from each other by the same amount (not necessarily the time delay between successive points in the original single variable data stream). This procedure is called "embedding the time series in a higher dimensional space." In this embedding space one can measure the relative separation of points to look for evidence of a fractal distribution.

The correlation dimension is the exponent that describes how the number of interpoint spacings less than some length epsilon grows with epsilon. Several examples of the plots of the correlation integrals versus epsilon for different signals measured from single mode lasers and from amplified spontaneous emission reveal the principal features of the technique.

The most important part of this process is the determination of the proper delay to use for the embedding and the proper number of points to use to gain an adequate representation of the attractor. It is increasingly clear that for low dimensional attractors which are visited relatively uniformly in time, a small number of points (such as 500-1000) may be adequate to estimate the properties of the attractor[28]. Several different methods for optimizing the process were proposed and discussed at a recent conference on the calculation of dimensions[29]. Calculation of the autocorrelation function or the mutual information function gives a correlation time which sets an appropriate time scale for the embedding

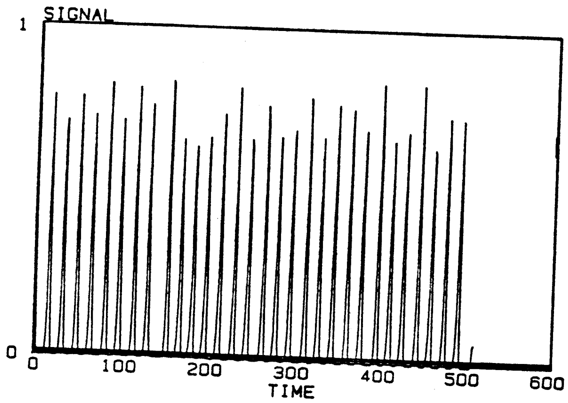

Figure 5a: Intensity vs. time signal for chaotic laser solution of equations.

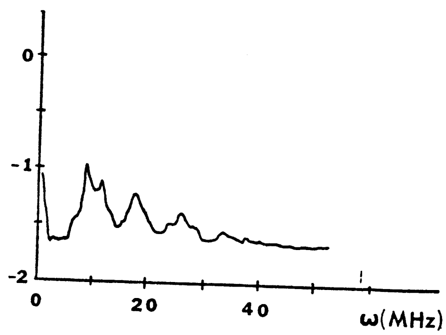

Figure 5c: Power spectrum of experimental laser signal.

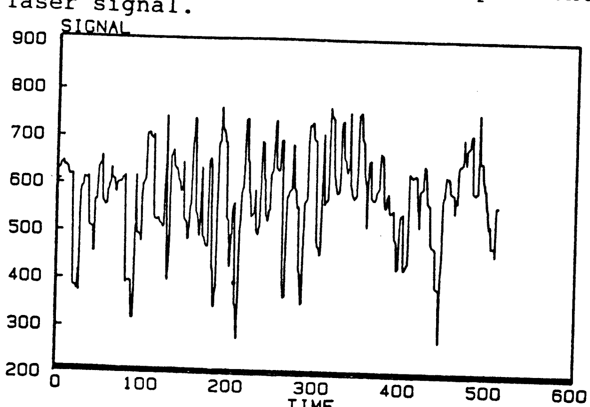

Figure 5e: Intensity vs. time signal for amplified spontaneous emission

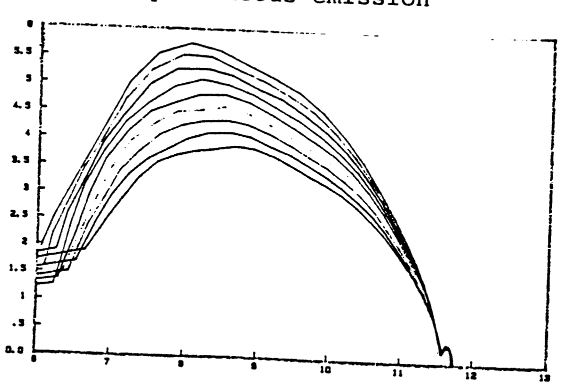

Figure 5g: Slope of correlation integral for digitized electronic amplifier noise.

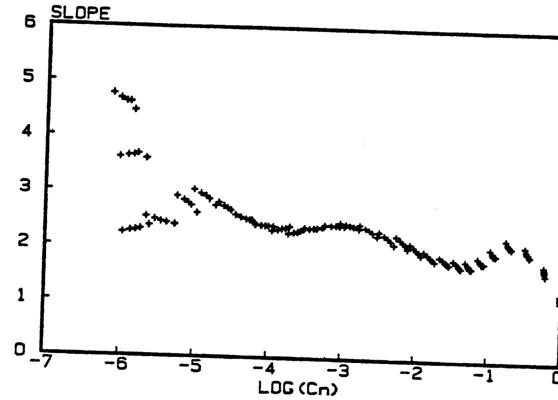

Figure 5b: Slope of correlation integral versus epsilon for dimensions 16-20. Plateau at 2.4±.2 indicates a fractal dimension.

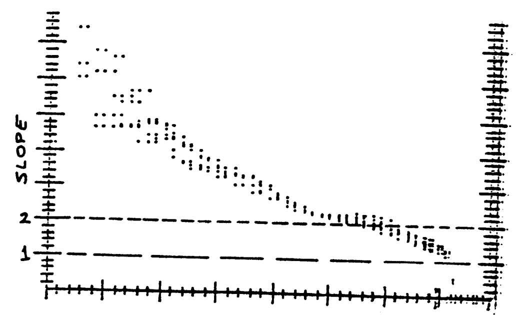

Figure 5d: Slope of correlation integral versus epsilon for dimensions 10-20. Plateau at 2.2±.2 indicates a fractal dimension.

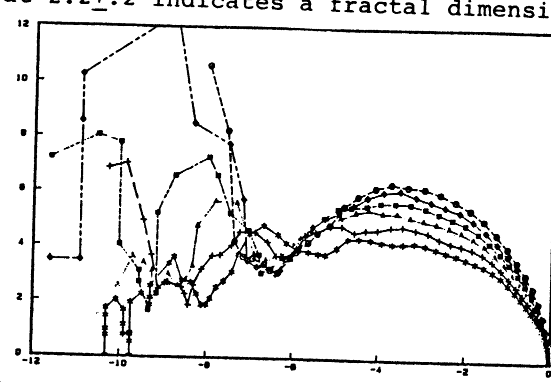

Figure 5f: Slope of correlation integral for ASE. No plateau indicates no difference between this signal and random noise.

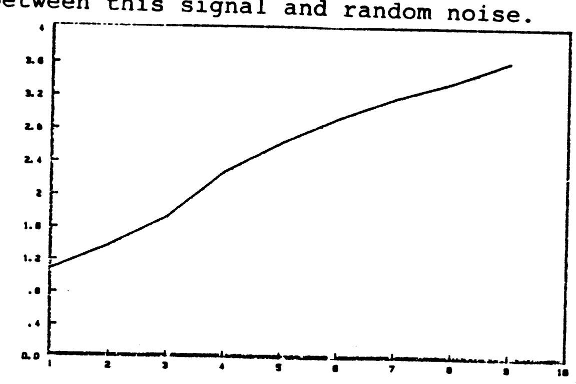

Figure 5h: Change in slopes for electronic noise with increasing embedding dimension.

vector[30]. It is generally necessary that each embedding vector involve delays of the order of the correlation time or the first minimum of the mutual information. In addition the number of components of the vector should be about twice the dimension. An improved method of determining an optimum embedding of the trajectory has been proposed by Broomhead and King[31]. By offering a suitable transformation of the normal delay embedding vectors, the B-K technique has been shown to greatly increase the efficiency of the calculation of dimensions and other features to be calculated from the data[32]. It also gives an embedding space of optimum and minimun size for measuring the deterministic nature of the solution.

One important note for those applying these techniques is the result that essentially all strange attractors have dimensions greater than or equal to two. This is because the fractal nature of the attractor is at least as big as a Cantor set transverse to a two-dimensional surface. Application of the quantitative measurement techniques often results in dimensions less than two. This most commonly happens when the measurements are made as an average of the distribution of the spacing of points on the attractor. If the attractor is twisted or compressed unevenly, the fractal nature can be apparent in one region on a particular length scale and be hidden in another region on the same length scale where, instead, only the basic one dimension of nearly similar trajectories is apparent. Technically, the fractal dimension is defined only in the limit of infinitesimal length scales to avoid this problem, but in many experimental situations it is impossible to approach infinitesimal spacings. In this case, we have found that the B-K technique also tends to find a basis set for the embedding that removes twists and compressions in a way that keeps the fractal nature on more nearly comparable length scales over the whole attractor. The result is a fractal dimension that is greater than two.

There are also conjectures about the relationship between the Lyapunov exponents of a system and the dimension of the attractor[33]. These have been used successfully in cases where the dimensionality is so high that its calculation by the Grassberger and Procaccia method will fail[34]. Then one must rely on computationally demanding methods for calculating the Lyapunov exponents[35].

Alternatives to Dimension Calculations

Several different shortcuts have been proposed for the measuring of properties of chaos or determining its presence. Most notable is the blurring of an oscilloscope trace repetitively triggered by initial conditions as accurately reset as possible[36]. Unfortunately this process is guaranteed only to offer information about the transient evolution of the system as it finds the attractor, but it appears that in some cases the transient is relatively innocuous and that traces represent in a reasonable way the divergence of trajectories with similar but slightly different initial values.

Others are actively measuring the coherence and statistical properties of chaos[37-39]. Probability distributions, moments and cumulants, correlation functions, and other such measures are popular in the study of statistics and have recently been studied for chaotic systems. It is not yet clear whether there are characteristics sufficiently unique to chaos to offer reliable proof of the presence of chaos.

Finally one can ask about the effect of noise on a chaotic system[40]. There are no absolute rules. In some cases, if the rms of the noise is smaller than the rms of the chaotic part of the signal about the basic structure of the attractor, it is possible that the noise blurs the fractal dimension on short length scales and does not significantly perturb the fractal on intermediate length scales. In principle, it is equally possible that the strange attractor is only very weakly stable with the result that a small amount of added noise can completely destroy the chaotic behavior. There are numerous instances where it appears that there is a fractal nature to the attractor over some length scales and that at shorter scales either the digitizing accuracy or inherent system noise causes blurring. By way of contrast, it is also possible that noise can disturb the stability of periodic or steady-state solutions causing the system to evolve in a chaotic way on a strange attractor that would be unstable in the absence of noise. This can be distinguished from "noise amplification" in that the noise-induced chaos has all the properties of a strange attractor while amplified noise would be more random.

As these and other quantitative measures are increasing applied to digitized signals, it will surely be true that more and more instances of irregular behavior will be explained as having their origins in dynamical evolution on strange attractors. Continued progress seems likely and simpler litmus tests involving new direct analog computations may emerge from current research efforts.

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