28 September 1986

Chaos Experiment: Quadratic Map

In this experiment we will use a computer to explore the *dynamics* of a simple looking *difference equation*

$$z_{n+1} = 4\lambda z_n(1 - z_n)$$

which is known both as the *quadratic* and *logistic map*. Although this equation looks simple, it exhibits some remarkably complex solutions. In fact, many of the properties of the quadratic map can only be discovered by computer simulations. The quadratic map is something like the simple harmonic oscillator of nonlinear dynamics. Many linear phenomenon in physics behave like a simple harmonic oscillator. Similarily, we shall see that many nonlinear phenomenon share qualitative and even quantitative properties with the quadratic map.

— Begin by reading the article **Universal Behavior in Nonlinear Systems** by Mitchell J. Feigenbaum [*Los Alamos Science* 1 4-27 (1980)] and working through the problems below in your lab book. You only need to read from the begining up to the section "Some Details of the Full Theory."

Problems

P1: Let $f(x) = 2x^2$. If $x_0 = 2$, then $x_1 = f^1(x_0) = 2(2)^2 = 8$, and $x_2 = f^2(x_0) = f^1(f^1(x_0)) = f^1(x_1) = 2(8)^2 = 128$. Calculate $f^4(x_0)$ when $x_0 = 0.5$. Let $f(x) = 2x$, calculate $f^4(x_0)$ when $x_0 = 2$.

P2: Let $f(x) = ax$, for an arbitrary x_0, calculate $f^4(x_0)$. In general, what does $f^n(x_0)$ equal?

P3: Let $f(x) = a - x^2$. Show that $f^2(x_0) = a - a^2 + 2ax_0^2 - x_0^4$.

P4: The difference equation $x_n = 4\lambda x_n(1 - x_n)$ is called the quadratic map. For $x_0 = 0.25$, 0.5, and 0.75 construct the following table:

$\lambda =$	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
0.25										
0.50										
0.75										
0.8										
0.9										

Write a simple calulator or computer program to find x_n if you think it will be quicker than doing it by hand. Explore the behavior of the quadratic map for other values of the initial condition x_0 and the parameter λ. Comment on any patterns you see in your data. How does the behavior of the quadratic map change as you vary the parameter λ. Does the high iterate solution ($\lim_{n\to\infty} x_n$) depend on the initial value of x_0.

P5: Calculate $f^8(x_0)$ where $f(x) = 3x(1 - x)$ and $x_0 = 0.25$ by the *graphical method*. Use graph paper and be sure to plot $f(x)$ between 0 and 1 accurately. Explore the quadratic map by the graphical method for several initial conditions and parameter values. What happens to points outside the unit interval [0,1] i.e., what happens to x_n when $x_0 > 1$ or $x_0 < 0$.

Programming

Below is an Applesoft BASIC program for iterating the logistic map. This can be run on Apples or Apple look-alikes. A is the parameter λ. The program can plot z_n as a function on A. Figure 1 is an output of this program for $0.73 < A < 0.93$ in steps of 0.002. It is called a *bifurcation diagram* since it shows how the asymptotic solutions (very large n's) of the map bifurcate from one point (period 1) to 2 points (period 2) at A_1, to 2^2 points at A_2, ..., to 2^k points before finally becoming chaotic at A_c.

```
10 INPUT "ENTER LOWER, UPPER LIMITS OF A:"; A1, A2
20 INPUT "ENTER STEP SIZE FOR A:"; DA
30 HGR
40 HCOLOR = 3
50 FOR A = A1 TO A2 STEP DA
60 XX = 250*(A - A1)/(A2 - A1)
70 X = 0.5
80 FOR I = 1 TO 200: X = 4*A*X*(1-X): NEXT I
90 FOR I = 1 TO 200: X = 4*A*X*(1-X)
100 YY = 179*(1-X)
110 HPLOT XX, YY
120 NEXT I
130 NEXT A
140 END
```

Program Notes

lines 30 and 40: HGR tells the computer to turn on its High Resolution Graphics mode. This converts all but the bottom four lines of the screen into a grid consisting of 280 points in the horizontal direction and 160 points in the vertical direction. You can command any one or any number of points (PIXELS in computerese) specified on this grid to light up by using the HPLOT command.

Each point on the grid is specified by its rectangular coordinate (X,Y) where the origin (0,0) is the top left corner; positive X is to the right and positive Y is downward.

HCOLOR = 3 specifies a color lever. HCOLOR = 0 means black, HCOLOR = 7 is brightest.

HGR AND THE SPECIFICATION OF HCOLOR MUST PRECEDE ANY GRAPHICS COMMANDS.

line 60: sets the scale for the x axis so you use all 280 (0 to 279) x-locations.

line 70: sets $x_0 = 0.5$ as the initial point.

line 80: throws out the first 200 iterates to get rid of transients.

line 100: sets scale for the y axis making use of 180 y-locations. Y still increases downward.

After the calculations and plotting are done, if you want to print your graph, type:

PR# 1 (RETURN)

[Control] I 2 H ("eye" two H).

You may "blow up" any horizontal section of the graph by putting in appropriate values of A1, A2, and DA (lines 10 and 20). You may change also the x scale as appropriate. If DA is not too small, you can determine the value of A by counting pixels on the graph.

Calculations

Use the above program or a program of your own to determine the values A1, A2, ..., of A at a few bifurcation points, and use these values to estimate,

$$\delta_f \approx \frac{(A_n - A_{n-1})}{(A_{n+1} - A_n)}.$$

How do your δ_n's compare with *Feigenbaum's number*,

$$\delta_f = \lim_{n\to\infty} \delta_n?$$

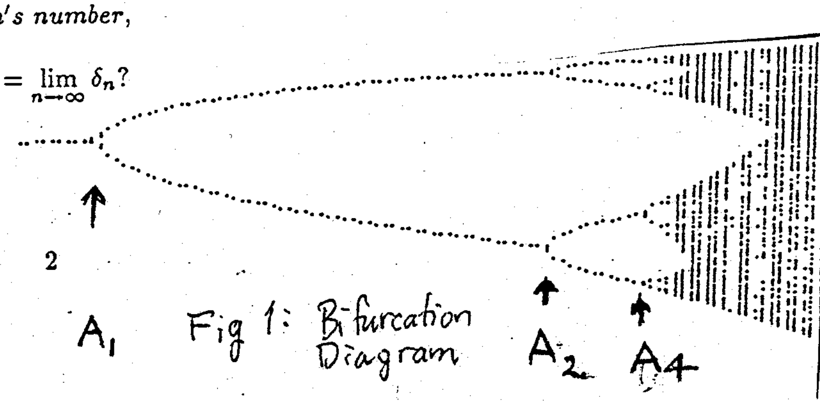

Fig 1: Bifurcation Diagram