Generating a fractal using a capacitor

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Digital communications systems often drive electronic circuits with signals that generate fractals. This paper describes a simple example, a triangular wave form executing a random walk applied to a circuit consisting of a capacitor and a resistor is used to generate a Cantor ''middle third'' set. This example can be used as part of a physics demonstration or an introductory lab.
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Examples of fractals arise in many areas of physics such as crystal growth (the formation of snowflakes) and chaotic solutions of n-body motions.[1] Probably the first mathematical example of a fractal is Cantor's middle third construction. Recall that this Cantor set is formed by starting with a fixed (one-dimensional) interval and erasing the ''middle third.'' This yields two subintervals from each of which the middle third is again erased. Applying this process repeatedly to each of the remaining subintervals leads to a fractal which is self-similar and has noninteger dimension.

To calculate the ''fractal dimension'' of the Cantor set note that the total length of the line segments at the nth iteration is $l_n=(\frac{2}{3})^n$, and the total number of line segments is $N_n=2^n$. So the length of each individual segment is $\epsilon_n=l/N=(\frac{1}{3})^n$. One measure of fractal dimension is the capacity dimension defined as $d_{\mathrm{cap}}=-\lim_{\epsilon\to 0}\ln N/\ln\epsilon=\lim_{n\to\infty} n\ln 2/n\ln 3=\ln 2/\ln 3$.[2]

A middle third Cantor set is easy to generate with what Barnsley calls an ''iterated function system.''[3] Specifically, consider a process of iteration that starts with an initial seed $x_0\in[-b,+b]$ and then calculates the next value x_1 by randomly picking one of two functions,

$$f_1(x)=\tfrac{1}{3}x-\tfrac{2}{3}b \tag{1}$$

or

$$f_2(x)=\tfrac{1}{3}x+\tfrac{2}{3}b. \tag{2}$$

This new value is then used as a seed to generate x_2, and so on. A moment's reflection shows that this ''random iteration'' process generates a Cantor middle third set on the interval $[-b,+b]$.[4] Later in this work we show how this random iteration process can be realized with a simple electronic circuit. The motivation for this example came from a talk by Broomhead in which he pointed out the role iterated function systems can play in modeling nonlinear signal channels.[5]

Consider a series RC circuit, as shown in Fig. 1, driven by a voltage source v_s. The voltage around a closed loop is

$$v_s-V_C-V_R=v_s-\frac{1}{C}\int_0^t i\,dt-Ri=0 \tag{3}$$

or, differentiating with respect to time,

$$\frac{di}{dt}+\frac{1}{RC}i=\frac{1}{R}\frac{dv_s}{dt}. \tag{4}$$

The voltage source is a triangular wave form defined as follows. For an integer n and a real time

$$\tau=t_{n+1}-t_n, \tag{5}$$

v_s is a linear function of the voltage with slope plus or minus one. The slope's sign randomly changes sign at each instant indexed by n. The waveform is thus a random walk; however, for practical implementations, additional constraints, such that the root-mean-square deviation of the drive signal is zero and that there is an upper bound on the absolute value of the voltage may be required. Such constraints are common for practical digital communications codes.[6] By construction,

$$\frac{dv_s}{dt}=s_n,\quad s_n\in\{-1,+1\}, \tag{6}$$

generates a random sequence of form

$$\{\ldots,+1,+1,-1,+1,-1,-1,-1,+1,\ldots\}.$$

This type of voltage source can be implemented with a programmable arbitrary wave form generator found, for instance, in an analog input/output card. Such multi-function cards fit into a PC's ISA or PCI expansion slots and are relatively inexpensive.

The equation of motion for the current is then

$$\frac{di}{dt}=\frac{-1}{RC}i+\frac{1}{R}s_n. \tag{7}$$

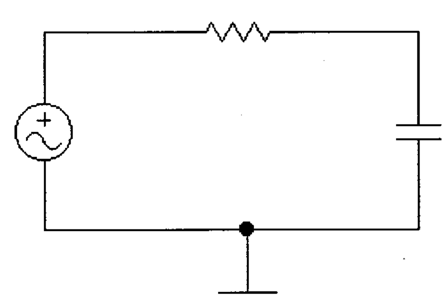

Fig. 1. Schematic for a series RC circuit.

Depending on the sign s_n, the equation contains two possible equilibrium points $(di_*/dt=0)$,

$$i_*=Cs_n, \quad s_n\in\{-1,+1\}. \tag{8}$$

When $t\neq t_n$, $n=0,1,2,...$, the equation of motion is easy to solve by considering the motion relative to the equilibrium point,

$$y=i-Cs_n; \tag{9}$$

then

$$\frac{dy}{dt}=\frac{-1}{RC}y \tag{10}$$

with solution

$$y(t)=y_0e^{-(1/RC)t}. \tag{11}$$

Next the current is found in the original coordinates at each $t_{n+1}[i(n\tau)=i_n]$,

$$i_{n+1}=e^{-(1/RC)\tau}i_n+C(1-e^{-(1/RC)\tau})s_n. \tag{12}$$

Now note that Eq. (12) is exactly in the form of Eq. (1) or (2), depending on the sign of s_n, when

$$e^{-(1/RC)\tau}=\tfrac{1}{3}. \tag{13}$$

This implies

$$i_{n+1}=\tfrac{1}{3}i_n+\tfrac{2}{3}Cs_n, \quad s_n\in\{-1,+1\}. \tag{14}$$

For instance, if $C\approx10^{-6}F$ and $R\approx10^4\Omega$, then $\tau\approx10^{-2}$ s or 10 ms, which is well within the range of most inexpensive analog input/output cards.

This brief note is meant to provide a simple example that illustrates the ubiquitous existence of fractals in modern digital communications systems and to point toward an interesting path for further studies which I hope will be undertaken by interested students of physics.

[1]Alan J. Hurd, ''Resource Letter FR-1: Fractals,'' Am. J. Phys. **56**, 969–975 (1988).

[2]Robert L. Devaney, *An Introduction to Chaotic Dynamical Systems* (Addison–Wesley, New York, 1989).

[3]Michael F. Barnsley, *Fractals Everywhere* (Morgan-Kaufmann, Los Altos, CA, 1993).

[4]S. Redner, ''Random multiplicative processes: An elementary tutorial,'' Am. J. Phys. **58**, 267–273 (1990).

[5]D. S. Broomhead, J. Huke, and M. Muldoon, ''Digital Channels,'' presented at SIAM's Pacific Rim Dynamical Systems Conference, 9–13 August 2000, Lahaina, Maui, Hawaii; D. S. Broomhead, J. Huke, and M. Muldoon, ''Fractals, Linear Channels and Delay Methods,'' preprint.

[6]C. C. Bissell and D. A. Chapman, *Digital Signal Transmission* (Cambridge U. P., New York, 1997).

THE UPSTART METER

The mètre derives its authority as a standard from a law of the French Republic in 1795.

It is defined to be the distance between the ends of a rod of platinum made by Borda, the rod being at the temperature of melting ice. This distance was chosen without reference to any former measures used in France. It was intended to be a universal and not a national measure, and was derived from Delambre and Mechain's measurement of the size of the earth. The distance measured along the earth's surface from the pole to the equator is nearly ten million of mètres. If, however, in the progress of geodesy, a different result should be obtained from that of Delambre, the mètre will not be altered, but the new result will be expressed in the old mètres. The authorised standard of length is therefore not the terrestrial globe, but Borda's platinum rod, which is much more likely to be accurately measured.

The value of the French system of measures does not depend so much on the absolute values of the units adopted as on the fact that all the units of the same kind are connected together by a decimal system of multiplication and division, so that the whole system, under the name of the metrical system, is rapidly gaining ground even in countries where the old national system has been carefully defined.

The mètre is 39.37079 British inches.

James Clerk Maxwell, *Theory of Heat* (Appleton, New York, 1872), pp. 77–8.