International Journal of Bifurcation and Chaos, Vol. 3, No. 3 (1993) 779–784

DISCRETE DYNAMICAL MODELS SHOWING PATTERN FORMATION IN SUBAQUEOUS BEDFORMS

NICHOLAS B. TUFILLARO
Center of Nonlinear Studies, MS-B258,
Los Alamos National Laboratory, Los Alamos, NM 87545, USA

Received August 13, 1992; Revised October 15, 1992

A new class of "toy models" for subaqueous bedform formation are proposed and examined. These models all show a similar mechanism of wavelength selection via bedform unification, and they may have applications to bedform stratigraphy. The models are also useful for exploring general issues of pattern formation and complexity in stochastically driven far-from-equilibrium systems.

1. Introduction

The patterns formed (ripples and dunes) on the face of a sandy bed beneath a flowing liquid (rivers or tides) are a beautiful example of self-organization in a geological system. Faithful models of geological pattern formation (geomorphology) can be difficult to specify because of the nonlinear properties of the material substances (multiphase flows) as well as the complex and time-dependent boundary conditions present in the environment. Indeed, on the face of it, it seems remarkable that highly organized rippled bedforms are found to exist with such great regularity over a wide range of length scales and in many diverse environmental settings [Allen, 1968; Hallet, 1990]. For instance, mud waves on the oceanic abyssal plane can be 200 meters high and spaced over 1 kilometer apart. Whereas the size scale for sand ripples found beneath tidal currents is of the order of millimeters.

Here I describe a new class of "toy models" which might be useful in explaining some of the self-organizational mechanisms occurring in the formation of subaqueous ripples and dunes. The cellular-automata-type models proposed are inspired by recent developments in physics (the theory of self-organized criticality [Bak & Chen, 1991; Bak *et al.*, 1988]) and geology (the formation of eolian ripples [Forrest & Haff, 1992]). This class of models could also lend itself to applications involving subaqueous bedform stratigraphy.

In Sec. 2 of this paper, I briefly review some features of pattern formation in subaqueous ripples and dunes. In Sec. 3, I provide a (perhaps naive) classification scheme for the microscopic mechanisms of sediment transport and then introduce a new class of cellular automata models to mimic these basic sediment transport mechanisms. In Sec. 4, results from simulations of some of the simpler sediment transport rules is presented and the common self-organizational principles exhibited by these models is described. Finally, in Sec. 5, I speculate on the relationship between these microscopic models and continuum (macroscopic) models for subaqueous bedform evolution.

2. Basic Subaqueous Bedform Patterns and Transitions

At least three basic bedforms (patterns) have been identified for fixed flow and sediment parameters: flat beds, ripples, and dunes. Several experimental studies [Southard, 1991] have also identified definite transitions between these basic states as some parameter in the flow-sediment system is varied, say the flow strength. As the flow strength is increased, the simplest observed scenario is a transition from

$$\text{flat bed (low flow velocity)} \rightarrow \text{ripples} \rightarrow \text{dunes} \rightarrow \text{flat bed (high flow velocity)}.$$

Ripples can occur at small flow strengths where the flow over the bedface is laminar. Ripples often exist as long straight rides oriented perpendicular to the flow, although more complex patterns are also common. The "ripple index" (that is, the average ripple wavelength over the average ripple height) as a rule of thumb tends to be of order 10. Ripple geometry is marked by gradual slopes on the upstream side of the flow and more steep slopes close to the angle of repose on the downstream side. Ripples (unlike dunes) are strongly nonhydrostatic, that is, the ripple geometry does not depend strongly on the flow depth. In laminar flow regimes, ripple dynamics appears to be dominated by bed load transport.

Dunes occur at higher flow strengths where turbulent eddies exist, particularly in the flow of the wake of a dune. Accordingly, sediment in suspension (suspended load) is an important transport mechanism. Dunes are strongly hydrostatic. In flume studies, dune height tends to be one-fifth the flow depth. The "dune index," as a rule of thumb, is of order 5. Dunes can have a more symmetric geometry than ripples. Like ripples they can be oriented in long straight ridges perpendicular to the flow, or exhibit far more complex patterns. Ripples and dunes can coexist in the same bedform, with smaller ripples traveling up the back of larger dunes.

The idea of forming an analogy between equilibrium thermodynamics phase diagrams and pattern dynamics is a recurring theme in the study of complex systems. In the context of subaqueous bedforms, this same point of view has been advocated by Southard and coworkers [Southard, 1991] who identify such fundamental parameters as flow speed, sediment size, and flow depth and then experimentally construct pattern dynamics "phase diagrams" showing the relationships between the different bedforms and the "pattern" parameters. Extensive analysis of flume and field studies shows that the pattern dynamics scenario described above is, at best, only a very rough approximation to the plethora of patterns and transitions that occur in a real flow-sediment system.

3. Discrete Microscopic Models of Sediment Transport

I find it convenient to separate the basic microscopic transport processes into three basic categories or mechanisms: bed load transport, avalanche transport, and suspended load transport. The last case, suspended load transport, consists of the dual processes of scouring and deposition. Usually, all these transport mechanisms occur only at a thin layer on the surface of the bedface, a few grain diameters thick.

Bed load transport is the local (relative to the bed surface), short-range movement of sediment particles tangential to the bedface. It consists of hopping or rolling motions of individual sediment grains. Typically there is a short transport distance, say a few grain diameters. This transport mechanism exists in both laminar and turbulent flows, and is driven by the sheer stress generated by the flow on individual sediment particles. Presumably, this is the dominant mode of sediment transport on the upstream side of bed features with a gradual slope.

Avalanche transport can occur on the down stream side of bed features with a steep slope close to the angle of repose. Avalanche processes are quite visible in experimental flume studies, and in some parameter regimes are clearly an important method of sediment transport. However, to the best of my knowledge, no previous studies have explicitly incorporated avalanche processes into the modeling of bedform features and sediment transport. I also think of the avalanche process as a local mechanism which transports sediment in sudden bursts. Much of the sediment transported is, again, tangential to the bedface. Although in a turbulent flow regime, avalanches can also act to lift large amounts sediment into suspension.

Suspended load transport occurs when turbulent eddies and bursts lift sediment grains into suspension (scouring). These sediment grains (in suspension) are then carried by the flow until they are redeposited (deposition) on the sediment bed. I think of suspended load transport as a nonlocal mechanism which transports sediment particles perpendicular to the bedface.

To model the microscopic dynamics of the bedface (fluid-sediment interface), I begin by considering a one-dimensional lattice $j = 1, 2, \ldots, L$ and associate a sediment height h_j to each site. For most of the models considered here, I also assume cyclic boundary conditions. The discrete slope at each site is defined by

$$s(j) = h(j) - h(j-1)\,. \tag{1}$$

At a discrete time step t, the number of particles at each site can change according to a sediment transport rule

$$h(j) = B(\cdot) + A(\cdot) + D(\cdot) - S(\cdot)\,, \tag{2}$$

where B is a rule describing the bed load transport, A the avalanche transport, S the suspended load transport due to scouring, and D the suspended load transport due to deposition. In general, of course, each of these rules can be fantastically complicated, and can depend on a wide range of variables such as the local and nonlocal sediment height (h), slope (s), an integrated surface area (I), and flow parameters such as the flow strength (Re) or sediment size (Fr). For instance, for a complete microscopic theory, I would need to solve the Navier–Stokes equations with time-dependent boundary conditions, and then within this flow solution simultaneously solve for the Newtonian motion of individual sediment particles subject to stresses, flow forces, gravitational force, buoyancy forces, and so on.

Our main point here, though, is to consider a drastically simpler class of toy models for the transport mechanisms in order to explore general features of pattern formation which I hope may hold in a much larger class of transport models.

To this end, I could, for instance, ignore the complications due to the flow field by assuming that the flow field (perhaps in a turbulent regime) stochastically drives the bedface. In this type of toy model, I might imagine that each site randomly ejects a sediment particle according to some transition probability p_u. Next, this suspended particle is transported a fixed horizontal distance with each time step. I could think of this adjustable distance as the flow parameter Re. The particle in the flow then gets deposited with some downward transition probability p_d back on the bedface. Meanwhile, the bedface itself can undergo dynamical surface processes such as avalanches. I like to think of this particular rule as a two-level model for sediment transport, the fluid level is pictured as a conveyor belt since it simply carries particles randomly ejected at site j to a downstream site $j + n (n \in Z^+)$ in a few time steps.

As a specific example, consider a stocastically driven, avalanche dominated, transport model where at each time step t, each site obeys the Kadanoff type avalanche rule [Kadanoff *et al.*, 1989],

$$A(\cdot) = \begin{cases} \text{if } s(i) < -2, & h(i) \to h(i) + 2 \quad \text{and } h(i-1) \to h(i-1) - 2\,, \\ \text{if } s(i) > +1, & h(i) \to h(i) - 1 \quad \text{and } h(i-1) \to h(i-1) + 1\,. \end{cases} \tag{3}$$

Additionally, I consider an initially unpopulated suspended sediment flow variable $f(j)$ which follows the translation rule at each time step t,

$$f(j) \to f(j + Re)\,. \tag{4}$$

The flow and height variables are coupled by the stochastic rule

$$S(\cdot) = \begin{cases} \text{if } (s(j) > 0 \text{ and } \mathrm{rand}(j,\, t) > p_u)\,, & h(j) \to h(j) - 1\,, \\ & f(j) \to f(j) + 1\,, \end{cases} \tag{5}$$

where $\mathrm{rand}(j,\, t)$ is a random variable between [0, 1] and p_u is a fixed (or variable) transition probability for particle ejection from the bedface, and

$$D(\cdot) = \begin{cases} \text{if } f(j) > 0\,, & f(j) \to f(j) - 1\,, \\ & h(j) \to h(j) + 1\,, \end{cases} \tag{6}$$

and D is the rule specifying the deposition process. In this framework a bedface "equation of motion" is of the form

$$s_j(t+1) - s_j(t) = A(h_{j-1},\, h_j) + \mathrm{rand}(h_1,\, h_2, \ldots,\, h_L,\, t,\, t+1)\,. \tag{7}$$

From a continuum limit, this bedface dynamic can be viewed as a kind of (one-sided) nonlinear diffusion process which is driven stochastically.

4. Simulations Showing Self-Organization

I have explored a large number of sediment transport rules of the type discussed in the previous section. By fine tuning these rules, it is possible to mimic the (possibly asymmetric) geometry of bedform features, such as the angle of an individual dune on the upstream and downstream side of the flow, and to inhibit dune growth and size (for instance, by making the transition probability dependent on the height and slope variables). The development of specific rules for given features could have important applications for studies of subaqueous ripple stratigraphy. A similar viewpoint was recently put forth by Forrest and Haff in the context of eolian ripples [Forrest & Haff, 1992].

However, our most interesting observation is that the basic pattern formation processes are not very sensitive to the specific rules chosen. Indeed, finite amplitude wavelength selection in all the models studied appears to be of the "bedform unification" type suggested by Raudkivi & Witte [1990].

This bedform unification process can be seen in Fig. 1 which shows the development of a rippled bedface from a initially flat bed for the avalanche dominated transport model described in Sec. 3. Due to the stochastic driving, an initially flat bed quickly develops small ripples. These ripples then propagate to the right with individual ripple speeds which decrease with the ripple size. Thus smaller ripples collide with and overtake bigger ripples. This collision process is quite interesting. If the size difference in the ripples is great, then the smaller ripple is absorbed by the bigger ripple in one pass. This absorption mechanism is a diffusive like process. The speed of the smaller ripple increases as it moves up the back of the bigger ripple, but its characteristic width also increases so that the smaller ripple appears to diffusively spread out until it is

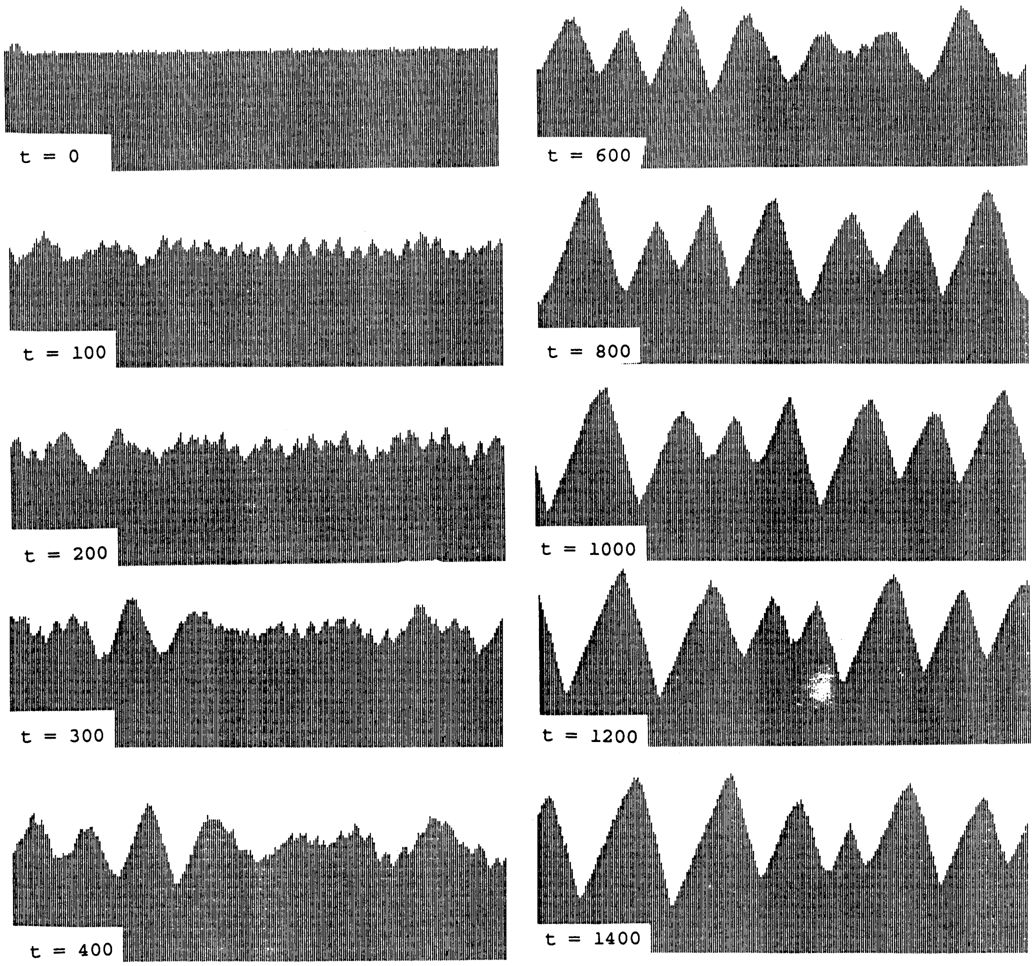

Fig. 1. Evolution and propagation of subaqeuous sand waves in our minimal model.

indistinguishable from the larger ripple. For ripples of a smaller size difference, the smaller ripple emerges from the collision with less mass, some of the mass of the smaller ripple having been absorbed by the larger rippler. The smaller ripples than continues for several more collisions until it is completely absorbed.

A second process is also observed for ripples of a similar size. Similar size ripples tend to experience a weak repulsive interaction. If a ripple is slightly smaller it will slowly try to overtake the larger ripple. However, this (slow) overtaking process involves a net mass transfer which can often be from the bigger to the smaller ripple. Thus the smaller ripple grows, slows down, and fails to overtake the initially bigger ripple.

These two types of ripple interactions seem to be the key elements present in all models showing the development of regular finite amplitude rippled bedforms from an initially flat bed.[1] Qualitative observations of a similar nature have also been made in experiments studying ripples formed in flumes [Costello & Southard, 1981].

5. Continuum Models Based on the Erosion Equation

A similar type of pattern selection process should also be observable in continuum models based on the erosion equation [Raudkivi, 1967]. To recall the form of the erosion equation, consider an equation for the continuity of sediment

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\mathbf{u}\varepsilon) = 0\,, \tag{8}$$

where $\mathbf{u}$ is the velocity field of the fluid (say, for instance, a unidirectional flow in the x direction) and ε is the sediment concentration field. Let x be the downstream horizontal spatial direction and z the vertical direction. In this analysis we will not consider the y direction, perpendicular to the flow. Further, let $\eta(x, t)$ be the bed height and $z_g(x, t)$ the top of the fluid relative to the bed floor. Within the bed $\mathbf{u} = 0$. Integrating the continuity equation in the vertical direction and using the Leibniz rule[2] to handle the boundary terms yields the equation

$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_\eta}\left[\frac{\partial}{\partial t}\int_\eta^{z_g} \varepsilon dz + \frac{\partial}{\partial x}\int_\eta^{z_g} \mathbf{u}\varepsilon dz\right], \tag{9}$$

where ε_η is the sediment concentration within the bed. If we let $\mathbf{q}_s = \int_\eta^{z_g} \mathbf{u}_s \varepsilon dz$ and ignore the sediment transfer with the water column, then we arrive at the erosion equation

$$\frac{\partial \eta}{\partial t} = -\frac{1}{\varepsilon_\eta}\frac{\partial q_s}{\partial x}\,. \tag{10}$$

Of course, applying the erosion equation to modeling the bedface dynamics requires that some functional relation be established between η and q_s. However, given this relationship, we notice that the erosion equation is a type of kinetic wave equation [Whitman, 1974] that applies, for instance, to supersonic shock waves in which a similar behavior of wave packet interaction is observed as that described in Sec. 4.

To explicitly see how these diffusive properties arise from the erosion equation, assume that $q_s = q_s(\eta, \eta_x)$ depends locally on the bed height and slope. Then we arrive at diffusive type equation of the form

$$\frac{\partial \eta}{\partial t} + u_s\frac{\partial \eta}{\partial x} = D_s\frac{\partial^2 \eta}{\partial x^2}\,, \tag{11}$$

where $u_s = \partial q_s/\partial \eta$ is the convective speed of ripples and $D_s = -\partial q_s/\partial \eta_x$ is the diffusion/amplification factor.

Thus, the discrete models can be viewed as very rough numerical approximations to models based on sediment continuity and it may be possible to understand their behavior by a proper application of kinetic wave theory [Whitman, 1974]. On the other hand, since the underlying grain size is discrete, we like to view the discrete models as no less fundamental than the continuum based models.

[1] Similar results have been observed in other types of models for eolian ripple development by Peter Haff and subaqeous ripple development by Brad Werner (private communications).

[2] Leibniz rule: If

$$u(x, t) = \int_{a(x,t)}^{b(x,t)} f(x, s, t)ds\,,$$

then

$$u_x(x, t) = \int_{a(x,t)}^{b(x,t)} \frac{\partial f}{\partial x}ds + f(x, b, t)\frac{\partial b}{\partial x} - f(x, a, t)\frac{\partial a}{\partial x}\,.$$

From a physical point of view, I also find these models intriguing because of the way they can be used to explore the processes by which microscopic disorder can create macroscopic order.

Acknowledgments

I acknowledge helpful discussions and correspondence with Chris Edwards, Peter Haff, A. Raudkivi, David Rubin, Roger Samelson, John Selker, Brad Werner, and Kurt Wiesenfeld.

References

Allen, J. R. L. [1968] *Current Ripples: Their relation to patterns of water and sediment motion* (North Holland Publishing Company, Amsterdam).

Bak, P., Tang, C. & Wiesenfeld, K. [1988] "Self-organized criticality," *Phys. Rev. Lett.* **38** (1), 364–374.

Bak, P. & Chen, K. [1991] "Self-organized criticality," Scientific American 45–53 (January).

Costello, W. R. & Southard, J. B. [1981] "Flume experiments on lower-flow-regime bed forms in course sand," *J. Sediment. Petrol.* **51**, 846–864.

Forrest, S. B. & Haff, P. K. [1992] "Mechanics of wind ripple stratigraphy," *Science* **255**, 1240–1243.

Hallet, B. [1990] "Spatial self-organization in geomorphology: From periodic bedforms and patterned ground to scale-invariant topography," *Earth-Science Reviews* **29**, 57–75.

Kadanoff, L. P., Nagel, S. R. & Zhou, S. [1989] "Scaling and universality in avalanches," *Phys. Rev.* **A39** (12), 6524–6537.

Raudkivi, A. J. [1967] *Loose Boundary Hydralics* (Pergamon Press, New York).

Raudkivi, A. J. & Witte, H.-H. [1990] "Development of bed features," *J. Hydraulic Eng.* **116** (9), 1063–1079.

Southard, J. B. [1991] "Experimental determination of bed-form stability," *Annu. Rev. Earth Planet Sci.* **19**, 423–55.

Whitman, G. B. [1974] *Linear and nonlinear waves* (John Wiley and Sons, New York).